International Journal of Information and Management Sciences
31 (2020), 99-121. DOI:10.6186/IJIMS.202006_31(2).0001

International Journal of
Information
and
Management
Sciences

Fuzzy Entropy Measure with an Applications in Decision Making Under Bipolar Fuzzy Environment based on TOPSIS Method

Vikas Arya and Satish Kumar

Maharishi Markandeshwar University

Abstract

In this paper, based on the concept of Havrda-Charvat-Tsallis entropy, fuzzy entropy measure is introduced in the setting of fuzzy set theory. The properties of the new fuzzy measure are investigated in a mathematical view point. Several examples are applied to illustrate the performance of the proposed fuzzy measure. Comparison with several existing entropies indicates that the proposed fuzzy information measure has a greater ability in discrimaniting different fuzzy sets. Lastly, the proposed fuzzy information measure is applied to the problem of MCDM (multi criteria decision making) based on TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) method under bipolar fuzzy environment. Two models are constructed to obtain the attribute weights in the cases that the information attribute weights is partially known and completely unknown. An example is employed to show the effectiveness of the new MCDM method.

Keywords: Renyi entropy, Shannon entropy, Tsallis entropy, fuzzy set, bi-polar fuzzy sets, TOPSIS.

1. Introduction

The concept of the fuzzy set developed by Zadeh [35] to model and process uncertain information in a much better way. By assigning the membership degree between 0 and 1 to elements with respect to a set, the fuzzy set can describe the state between “belong to” and “not belong to”. Therefore, many kinds of uncertainty that cannot be depicted by classical sets can be well-described by fuzzy sets. Since its inception, fuzzy set theory has been applied in many areas such as automatic control, pattern recognition, decision-making etc. The entropy of a fuzzy set was first proposed by Zadeh [36] to depict the fuzziness. Following, Zadeh’s work, De Luca and Termini [7] proposed a probabilistic entropy measure for FSs (fuzzy sets). Yager [34] proposed an entropy measure of a fuzzy set regarding an absence of qualification between fuzzy set and its complement. They also put forward some axiomatic properties for the fuzzy entropy measure, according to which fuzzy entropy can be defined. Yager’s concept was extended by Higashi and klir [12] to a more general kind of fuzzy complementation. Because of its improtance in depicting a fuzzy set, the entropy measure of fuzzy sets has been developing to an active topic in fuzzy set theory. In the fuzzy environment, many entropy measure had been defined by various authors (see Bhandari and Pal [5], Hooda [9], Hwang and yung [11], Joshi and Kumar [14], Joshi and Kumar [16], Kosko [19], Joshi and Kumar [20], Li and Lu [21], Pal and Pal [24], Pal and Pal [25], Verma and Sharma [30]).

One parametric generalization of entropy function (see Shannon [27]) by Renyi [26] has achieved the attention of researchers worldwide. With the rapid advancement, the role of parameters in an information measure came into the notice of authors. The presence of parameters play an important role in an information measure as they make it more flexible from application point of view and enhances its scope of application. For instance, in a problem based on environmental issues, the different parameters may characterize the different environmental factors like as temperature, time constraint and pressure etc. More the number of parameters an information measure contains, more will be its utility in various applications. Therefore, there is a need to develop more and more effective measures to cover simple as well as complicated situations. This work is a sincere effort in this direction. In this paper, a new one parametric generalization of Shannon entropy for FSs is proposed. The proposed measure is an extension of Havrda-Charvat-Tsallis entropy which is neither additive nor non-additive entropy measure from probabilistic settings to fuzzy set theory. The generalized entropy theory has its own advantages. When the parameters change, the entropy value changes. Also, when the parameters change, the entropy becomes another entropy. The parameters have practically significance for entropy. In this manner how the parameters influence /affect the entropy is an interesting topic to study. Some researchers studied the generalized entropy (see Bhandari and Pal [5], Kumar et al. [14], Joshi and Kumar [16], Joshi and Kumar [17], Kumar and Kumar [20], Mishra and Rani [23], Xiong et al. [33]).

Bipolar fuzzy set (BFS) theory formalizes a unified approach to fuzziness and polarity. It also includes a basis for multiagent decision analysis and bipolar cognitive modeling and captures double sided (positive and negative or effect and side effect) nature of cognition and human perception. For example, when we want to express effect and side effect of a drug, we can use bipolar fuzzy valuations. Because side effect is a negative effect. In fuzzy set, we can not model negative effect because it has only membership degree of an element. So, bipolar fuzzy and fuzzy theory is different in terms of modeling of problems.

Decision making means that the best alternaive is selected from a finite set of feasible alternatives according to the multiple criteria. Decision making theory is an important branch, which is mostly in human activities. Because decision making problem are frequentely produced from a complicated environment, evalutaed information is usely fuzzy. In general, the fuzzy information takes two forms that is qualitative and quantity. The quantity fuzzy information can be expressed by fuzzy sets. Fuzzy set theory proposed by Zadeh [35] has been describe fuzzy quantity information which contains only a membership degree. Due to its successful application and some short comings, many researchers introduce the extended form of fuzzy set.

TOPSIS is the most implemented technique for decision-making problems which was introduced by Hwang and Yoon [10] especially in economics, medical sciences, social sciences, enginerring etc. The fusion between MCDM and FS theory has led to new decision theory, named as fuzzy multi-criteria decision making (FMCDM), where we have decision-maker models that can deal with uncertain knowledge and information. The most important thing is that, when we want to assess, judge or decide, we usually a natural language in which the words don't have a clear, definite meaning. As a result, we need fuzzy number to represent linguistic variables, to express the subjective judgement of a decision maker in a quantitive manner. As a generalization of the concept of the classical set, fuzzy set, intuitionistic fuzzy sets etc., Zhang [37] firstly extended the concept of bipolar fuzzy sets as an extension of FSs whose membership degree range is $[-1, 1]$. In the case of BFSs, every element having two membership values. One lies in the interval $[0, 1]$ and other lies in the interval $[-1, 0]$. Actually, a wide variety of human decision making is based on double-sided or bipolar judgmental thinking on a positive side and a negative side. For instance, common interests and conflict of interests, feedback and feedforward, hostility and friendship, effect and side effect, likelihood and unlikelihood and so forth are often the two sides in coordination and decision. Similarly, in the traditional Chinese medicine (TCM), "yang" and "yin" are the two sides. Yang is the positive and masculine side of a system and yin is the negative and feimine side of a system. The coexistence, equilibrium, and harmony of the two sides are considered a key for the mental and physical health of a person as well as for the stability and prosperity of a social system. Thus, BFSs indeed have potential impacts on many fields, including economics, artificial intelligence, computer science, information science, cognitive science, management science, decision science, medical science, social science and quantum computing. In recent years bipolar fuzzy sets seem to have been studied and applied a bit enthusiastically and increasingly. From the above discussion, the importance of FSs and TOPSIS method can be easily judged. However, a lot of research has been done on solving MCDM problems using fuzzy sets and intuitionistic fuzzy sets, but a very little research has been done on solving MCDM problems where ratings of alternatives are expressed by using bipolar FSs. This study is a sequel in this direction. The prime aims of introducing this study are: (1) To introduce a parametric fuzzy information measure based on the Havrda-Charvat-Tsallis entropy. (2) The justification of a flexible parameter is also given in the form of a senstitive analysis. (3) To introduce a new MCDM method based on the proposed measure and bipolar fuzzy TOPSIS method. To do so, the present study is managed as follows.

The remaining of the paper is structured as follows. Section 2 recalls some basic concepts and definitions related to FSs. In Section 3, we proposed a new fuzzy information measure and prove some basic properties. Section 4 presented the comparative study of the proposed measure with some existing fuzzy entropy measures. In Section 5, we extended the application of extended TOPSIS method with the help of examples in bipolar fuzzy environment. At last, the paper is concluded with "Conclusions" in Section 6.

2. Basic Concepts and Definitions

The following notioins are used in this section. $X = \{q_1, q_2, \ldots, q_k\}$ be a fixed set; $FSs(X)$ is the class of all fuzzy sets of X.

Definition 2.1. A fuzzy set M on X is described as :

$$M = \big\{\big(q, \mu_M(q)\big) \mid q \in X\big\}, \tag{2.1}$$

in which $\mu_M : X \to [0,1]$ is a membership function of M in FSs.

Definition 2.2. A fuzzy set $\tilde{M}$ is said to be crisper than M if

$$\mu_{\tilde{M}}(q) \leq \mu_M(q), \text{ when } \mu_M(q) \leq 0.5;$$

and

$$\mu_{\tilde{M}}(q) \geq \mu_M(q), \text{ when } \mu_M(q) \geq 0.5.$$

Definition 2.3. Let $M, N \in FSs(X)$ be such that

(a) $\overline{M} = \{(q, 1 - \mu_M(q)) \mid q \in X\}$. (Complement)

(b) $M \cup N = \{(q, \sup(\mu_M(q)), \mu_N(q)) \mid q \in X\}$. (Union)

(c) $M \cap N = \{(q, \inf(\mu_M(q)), \mu_N(q)) \mid q \in X\}$. (Intersection)

First time to measure the uncertainty degree associated with a fuzzy set was made by Zadeh [35], who defined the (weighted) entropy of a fuzzy set M with respect to set X in a much better way as:

$$H(M) = -\sum_{i=1}^{k} \mu_M(q_i)(q_i)\log(q_i), \quad q_i \in X. \tag{2.2}$$

De Luca and Termini [7] first gave the following axioms for entropy of FSs as :

A1 (Sharpness): $H(M)$ is minimum if and only if M is crisp set .

A2 (Maximality): $H(M)$ is maximum if and only if M is the most fuzzy set.

A3 (Resolution): $H(M) \geq H(M^*)$, if M^* is crisper than M.

A4 (Symmetry): $H(M) \geq H(\overline{M})$ where $\overline{M}$ is the complement set of M, i.e., $\mu_{\overline{M}}(q_i) = 1 - \mu_M(q_i)$.

Throughout this paper, it is assumed that $k \in I^+$ (set of positive integers) and all logarithms are to base $D = 2$.

Corresponding to Shannon entropy, De Luca and Termini [7] defined a fuzzy entropy for a fuzzy set M as :

$$H(M) = -\frac{1}{k}\sum_{i=1}^{k}[\mu_M(q_i)\log(\mu_M(q_i)) + (1 - \mu_M(q_i))\log(1 - \mu_M(q_i))]. \tag{2.3}$$

Later on Bhandari and Pal [5] suggested some new measures of fuzzy entropy. Corresponding to Renyi entropy [26] they defined :

$$H^{\alpha}(M) = \frac{1}{1-\alpha}\sum_{i=1}^{k}\log[\mu_M(q_i)^{\alpha} + \big(1-\mu_M(q_i)\big)^{\alpha}], \tag{2.4}$$

where $\alpha > 0(\neq 1)$.

A parametric fuzzy entropy can not be suitable for different situtaions of same nature. So, there is always a scope for the development of a new fuzzy entropy. In this paper, we proposed a new entropy of fuzzy sets. A comparison is made with some existing entropies to show the effectiveness of the proposed one.

3. A New Fuzzy Information Measure

For this section, we briefly review the theoritical concept of information theory and then introduce the Havrda-Charvat-Tsallis entropy, along with studing with their properties. Let $\Gamma_k = \{S = (s_1, s_2, \ldots, s_k) : s_i \geq 0; \sum_{i=1}^{k} s_i = 1\}$, $k \geq 2$ be set of k-complete probability distributions. For any probability distributions $S = (s_1, s_2, \ldots, s_k) \in \Gamma_k$, Shannon [27] defined an entropy as:

$$H_{\text{Shannon}}(S) = -\sum_{i=1}^{k}(s_i)\log(s_i). \tag{3.1}$$

Tsallis [29] introduced a generalized form of Shannon entropy, Tsallis entropy is defined by

$$H^{\alpha}_{\text{Tsallis}}(S) = \frac{1}{\alpha-1}\Big[1-\sum_{i=1}^{k} s_i^{\alpha}\Big]; \alpha \in (0,1)\cup(1,\infty). \tag{3.2}$$

Since, $\lim_{\alpha\to 1} H^{\alpha}_{\text{Tsallis}}(S) = H_{\text{Shannon}}(S)$.

In particular, Tsallis [29] and Renyi entropy [26] having a close relationship between them as follows:

$$H^{\alpha}_{\text{Renyi}}(S) = \frac{1}{\alpha-1}\log_D\Big(1-(1-\alpha)H_{\alpha}(S)\Big) = \frac{1}{\alpha-1}\log_D\sum_{i=1}^{k} s_i^{\alpha}, \tag{3.3}$$

where $H^{\alpha}_{\text{Renyi}}(S)$ is the Renyi entropy.

However, a major difference existes, the Renyi [26] and Shannon [27] entropy are additive whereas the Tsallis entropy [29] is non-additive.

$$\text{i.e., } H^{\alpha}_{\text{Tsallis}}(S,T) = H^{\alpha}_{\text{Tsallis}}(S) + H^{\alpha}_{\text{Tsallis}}(T) + (1-\alpha)H^{\alpha}_{\text{Tsallis}}(S)H^{\alpha}_{\text{Tsallis}}(T), \tag{3.4}$$

where $S, T \in \Gamma_k$.

Remark.

1. If $\alpha = 2$, (3.2) recovers Ginni-Simpson's index :

$$\text{i.e., } H^{\alpha=2}_{\text{Tsallis}}(S) = \Big[1 - \sum_{i=1}^{k} s_i^2\Big]. \tag{3.5}$$

2. If $\alpha = 2$, (3.3) recovers Renyi Index :

$$\text{i.e., } H^{\alpha=2}_{\text{Renyi}}(S) = \log_D \Big[\sum_{i=1}^{k} s_i^2\Big]^{-1}. \tag{3.6}$$

However, in the literature of information theory, there exists various generalizations of Shannon's entropy [27], we introduced a new information measure $H^{\text{new}}_{\alpha} : \Gamma_k \to \Re^+$ (set of positive real numbers); $k \geq 2$ as follows:

$$H^{\text{new}}_{\alpha}(S) = \frac{1}{\alpha - \alpha^{-1}} \sum_{i=1}^{k} (s_i^{\alpha^{-1}} - s_i^{\alpha});\ \alpha \in (0, 1) \cup (1, \infty). \tag{3.7}$$

Particular cases:

1. $H^{\text{new}}_{\alpha}(S) = H^{\text{new}}_{\alpha^{-1}}(S)$.
2. If $\alpha \to 1$, (3.7) recovers the Shannon [27] entropy.
3. If $s_1 = s_2 = \cdots = s_k = \frac{1}{k}$, $H^{\text{new}}_{\alpha}(S) = \frac{1}{\alpha - \alpha^{-1}}[k^{1-\alpha^{-1}} - k^{1-\alpha}]$ is an upper bound for $H^{\text{new}}_{\alpha}(S)$.
4. If $\alpha = 2$, (3.7) becomes

$$H^{\text{new}}_{\alpha=2}(S) = \frac{2}{3}\Big[\sum_{i=1}^{k} (\sqrt{s_i} - s_i^2\Big],$$

which is an interesting entropy for probability distribution $S = (s_1, s_2, \ldots, s_k)$.

3.1. Relation between proposed entropy and existing entropies:

1. Setting $\alpha^{-1} = \beta$, (3.7) becomes Sharma and Taneja [28] entropy.

$$\text{i.e., } \quad H^{\text{new}}_{(\alpha, \alpha^{-1}=\beta)}(S) = \frac{1}{\alpha - \beta}\Big[\sum_{i=1}^{k} (s_i^{\beta} - s_i^{\alpha})\Big].$$

2. $H^{\text{new}}_{\alpha}(S) = \dfrac{1}{\alpha - \alpha^{-1}}\Big[\exp\Big((1 - \alpha^{-1})H^{\alpha^{-1}}_{\text{Renyi}}(S)\Big) - \exp\Big((1 - \alpha)H^{\alpha}_{\text{Renyi}}(S)\Big)\Big]$, where $H^{\alpha^{-1}}_{\text{Renyi}}(S) = \dfrac{1}{1 - \alpha^{-1}} \log \sum_{i=1}^{k} s_i^{\alpha^{-1}}$ which is a close relation between Renyi [26] and proposed entropy.

3. $H_{\alpha}^{\text{new}}(S) = \dfrac{1}{\alpha-\alpha^{-1}}\Big[1-(\alpha^{-1}-1)H_{\text{Tsallis}}^{\alpha^{-1}}(S)-\Big[1-(\alpha-1)H_{\text{Tsallis}}^{\alpha}(S)\Big]\Big]$, where $H_{\text{Tsallis}}^{\alpha^{-1}}(S) = \dfrac{1}{\alpha^{-1}-1}\Big[1-\sum_{i=1}^{k} s_i^{\alpha^{-1}}\Big]$ which is a relation between Tsallis [29] and proposed entropy.
4. $H_{\alpha}^{\text{new}}(S) = \dfrac{1}{\alpha-\alpha^{-1}}\Big[1-(2^{1-\alpha^{-1}}-1)H_{\text{H-Ch}}^{\alpha^{-1}}\Big]-\Big[1-(2^{1-\alpha}-1)H_{\text{H-Ch}}^{\alpha}\Big]$, where $H_{\text{H-Ch}}^{\alpha^{-1}}(S) = \dfrac{1}{2^{1-\alpha^{-1}}-1}\Big[1-\sum_{i=1}^{k} s_i^{\alpha^{-1}}\Big]$ which is a relation between Havrda and Charvat [8], Darocozy [1] and proposed entropy.
5. $H_{\alpha}^{\text{new}}(S) = \dfrac{1}{\alpha-\alpha^{-1}}\Big[\Big(1-\dfrac{\alpha^{-1}}{\alpha^{-1}-1}H_{\alpha\text{-norm}}^{\alpha^{-1}}(S)\Big)^{\alpha^{-1}}-\Big(1-\dfrac{\alpha}{\alpha-1}H_{\alpha\text{-norm}}^{\alpha}(S)\Big)^{\alpha}\Big]$, where $H_{\alpha^{-1}\text{-norm}}(S) = \dfrac{\alpha^{-1}}{\alpha^{-1}-1}\Big[1-\Big(1-\sum_{i=1}^{n} s_i^{\alpha^{-1}}\Big)^{\alpha}\Big]$ and $H_{\alpha\text{-norm}}(S) = \dfrac{\alpha}{\alpha-1}\Big[1-\Big(\sum_{i=1}^{n} s_i^{\alpha}\Big)^{\alpha^{-1}}\Big]$ which is an entropy studied by Arimoto [2] and Boekee- Lubbe [4]. Therefore, it becomes a close relationship between proposed entropy and Arimoto [2], Boekee-Lubee [4] entropy.

For some $S \in \Gamma_k$ we state the above entropy (3.7), as given below :

$$H_{\alpha}^{\text{new}}(S) = \frac{1}{2^{1-\alpha^{-1}}-2^{1-\alpha}}\left[\sum_{i=1}^{k}(s_i^{\alpha^{-1}}-s_i^{\alpha})\right];\ \alpha \in (0,1)\cup(1,\infty). \tag{3.8}$$

(3.7) and (3.8) are called a joint representatition of Havrda-Charvat-Tsallis entropy and essentially have the same expression except the normalized factor. The entropy (3.8) is normalized to one. That is, if $S = \left(\frac{1}{2}, \frac{1}{2}\right)$, the entropy (3.8) is one, whereas the entropy (3.7) is not normalized.

3.2. Properties of parametric entropy

Theorem 3.1. *The parametric entropy $H_{\alpha}^{\text{new}}(S)$, $S \in \Gamma_k$ has the following properties:*

1. ***Symmetry*** : $H_{\alpha}^{\text{new}}(s_1, s_2, \ldots, s_k)$ *is a symmetric function of* $(s_1, s_2, \ldots, s_k)$.
2. ***Non-Negative*** : $H_{\alpha}^{\text{new}}(S) \geq 0$ *for all* $\alpha > 0 (\neq 1)$.
3. ***Expansible*** : $H_{\alpha}^{\text{new}}(s_1, s_2, \ldots, s_k, 0) = H_{\alpha}^{\text{new}}(s_1, s_2, \ldots, s_k)$.
4. ***Decisive*** : $H_{\alpha}^{\text{new}}(0,1) = 0 = H_{\alpha}^{\text{new}}(1,0)$.
5. ***Maximility*** :
$$H_{\alpha}^{\text{new}}(s_1, s_2, \ldots, s_k) \leq H_{\alpha}^{\text{new}}\Big(\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k}\Big) = \frac{1}{\alpha-\alpha^{-1}}\Big[k^{1-\alpha^{-1}}-k^{1-\alpha}\Big].$$
6. ***Concavity*** :
$$H_{\alpha}^{\text{new}}(tS_1+(1-t)S_2) \geq tH_{\alpha}^{\text{new}}(S_1)+(1-t)H_{\alpha}^{\text{new}}(S_2).$$

7. ***Continuity*** : *$H_\alpha^{\text{new}}(s_1, s_2, \ldots, s_k, 0)$ is continuous in the region $s_i \geq 0$ for all $i = 1, 2, \ldots, k$ and $\alpha > 0$.*

Proof. Proof of the above theorm are trivial and omitted.

3.3. Definition

Corresponding to (3.7), we proposed the following fuzzy information measure:

$$H_\alpha^{\text{new}}(M) = \frac{1}{k(\alpha - \alpha^{-1})} \sum_{i=1}^{k} \Big[\big(\mu_M(q_i)\big)^{\alpha^{-1}} + \big(1 - \mu_M(q_i)\big)^{\alpha^{-1}} - \big(\mu_M(q_i)\big)^{\alpha} + \big(1 - \mu_M(q_i)\big)^{\alpha} \Big]. \tag{3.9}$$

Remark. If $\alpha \to 1$, (3.9) recovers the fuzzy measure (2.3).

Next theorem gives the validity of the proposed measure (3.9).

Theorem 3.2. *The measure* (3.9) *is a valid fuzzy measure.*

Proof. For validity the measure defined by (3.9), we should fulfill the axiomatic requirements (A1)−(A4).

A1 (Sharpness): From (3.7), we have

$$H_\alpha^{\text{new}}(M) = \frac{1}{k(\alpha - \alpha^{-1})} \sum_{i=1}^{k} \mu_M(q_i)^{\alpha^{-1}} + \big(1 - \mu_M(q_i)\big)^{\alpha^{-1}} - \Big(\big(\mu_M(q_i)\big)^{\alpha} + \big(1 - \mu_M(q_i)\big)^{\alpha} \Big). \tag{3.10}$$

If $H_\alpha^{\text{new}}(M) = 0$ in (3.7), then

$$\mu_M(q_i)^{\alpha^{-1}} + \big(1 - \mu_M(q_i)\big)^{\alpha^{-1}} - \Big(\big(\mu_M(q_i)\big)^{\alpha} + \big(1 - \mu_M(q_i)\big)^{\alpha} \Big), \text{ for all } i = 1, 2, \ldots, k. \tag{3.11}$$

Since $\alpha > 0$, (3.11) is satisfied only if $\mu_M(q_i) = 0$ or 1, for all $i = 1, 2, \ldots, k$. Conversely, let M be a non fuzzy set, i.e., crisp set, then either $\mu_M(q_i) = 0$ or 1. This implies that

$$\mu_M(q_i)^{\alpha^{-1}} + \big(1 - \mu_M(q_i)\big)^{\alpha^{-1}} - \Big(\big(\mu_M(q_i)\big)^{\alpha} + \big(1 - \mu_M(q_i)\big)^{\alpha} \Big), \text{ for all } i = 1, 2, \ldots, k. \tag{3.12}$$

and $\alpha > 0$.

Hence, $H_\alpha^{\text{new}}(M) = 0$ iff M is a crisp set, i.e., $\mu_M(q_i) = 0$ or 1 for all $i = 1, 2, \ldots, k$

A2 (Maximality): Differentiating (3.7) with respect to $\mu_M(q_i)$, we get

$$\begin{aligned} \frac{\partial H_\alpha^{\text{new}}(M)}{\partial \mu_M(q_i)} = & \frac{1}{k(\alpha - \alpha^{-1})} \Bigg[\frac{1}{\alpha} \Big\{ \mu_M(q_i)^{\frac{1-\alpha}{\alpha}} - \big(1 - \mu_M(q_i)\big)^{\frac{1-\alpha}{\alpha}} \Big\} \Bigg] \\ & - \Big[\alpha \Big\{ \mu_M(q_i)^{\alpha^{-1}} - \big(1 - \mu_M(q_i)\big)^{\alpha} \Big\} \Big]. \end{aligned} \tag{3.13}$$

Differentiating (3.13) with respect to $\mu_M(q_i)$ again, we get,

$$\frac{\partial^2 H_\alpha^{\text{new}}(M)}{\partial \mu_M(q_i)^2} = \frac{\alpha}{k(\alpha^2-1)}\Bigg[\Big(\frac{1}{\alpha^2}-\frac{1}{\alpha}\Big)\Big(\mu_M(q_i)^{\frac{1}{\alpha}-2} + \Big(\big(1-\mu_M(q_i)\big)^{\frac{1}{\alpha}-2}\Big)\Big)\Bigg] - \Big[(\alpha^2-\alpha)\Big(\mu_M(q_i)^{\alpha-2} + \big(1-\mu_M(q_i)^{\alpha-2}\big)\Big)\Big]. \tag{3.14}$$

Case (i). When $\alpha < 1$.

Let

$$A_1 = \Big(\frac{1}{\alpha^2}-\frac{1}{\alpha}\Big)\mu_M(q_i)^{\frac{1}{\alpha}-2} + \big(1-\mu_M(q_i)\big)^{\frac{1}{\alpha}-2},$$

$$A_2 = (\alpha^2-\alpha)\mu_M(q_i)^{\alpha-2} + \big(1-\mu_M(q_i)\big)^{\alpha-2},$$

Now $A_1 < 0$, $A_2 > 0$ and $|A_1| \le |A_2|$ for $\alpha < 1$, and $A_1 > 0$, $A_2 < 0$ and $|A_2| < |A_1|$ for $0 < \alpha < 1$. Also $\frac{1}{\alpha} > 1$, $\frac{\alpha^2-1}{\alpha} < 0$ for $\alpha < 1$. This implies that $\frac{\partial^2 H_\alpha^{\text{new}}(M)}{\partial \mu_M(q_i)^2} < 0$.

Case(ii).

Similarly, we can prove $\frac{\partial^2 H_\alpha^{\text{new}}(M)}{\partial \mu_M(q_i)^2} < 0$ for $\alpha > 1$. It is evident that

$$\frac{\partial H_\alpha^{\text{new}}(M)}{\partial \mu_M(q_i)} = 0, \quad \text{when } \mu_A(q_i) = 0.5.$$

This shows that $H_\alpha^{\text{new}}(M)$ is a concave function and has a global maximum at $\mu_a(q_i) = 0.5$. It proves that $H_\alpha^{\text{new}}(M)$ is maximum if and only if M is the most fuzzy set, i.e., $\mu_M(q) = 0.5$ for all q.

A3 (Resolution): In (3.13), for all $\alpha > 1$

$$\frac{\partial H_\alpha^{\text{new}}(M)}{\partial \mu_M(q_i)} > 0, \quad \text{in } [0, 0.5)$$

and for $\alpha < 1$

,

$$\frac{\partial H_\alpha^{\text{new}}(M)}{\partial \mu_M(q_i)} < 0, \quad \text{in } (0.5, 1]$$

and $\frac{\partial H_\alpha^{\text{new}}(M)}{\partial \mu_M(q_i)} = 0$ at $\mu_M(q_i) = .5$.

Therefore, $H_\alpha^{\text{new}}(M)$ is an increasing function of $\mu_M(q_i)$ in $[0, 0.5)$ and decreasing function of $\mu_M(q_i)$ in $(0.5, 1]$.

Now, let M^* be crisper than M. This implies

$$0 \le \mu_{M^*}(q_i) \le \mu_M(q_i) < 0.5 \Rightarrow H_\alpha^{\text{new}}(M^*) \le H_\alpha^{\text{new}}(M) \tag{3.15}$$

and

$$0.5 < \mu_M(q_i) \le \mu_{M^*}(q_i) \le 1 \;\Rightarrow\; H_\alpha^{\text{new}}(M^*) \le H_\alpha^{\text{new}}(M). \tag{3.16}$$

From (3.15) and (3.16), we get $H_\alpha^{\text{new}}(M^*) \le H_\alpha^{\text{new}}(M)$ where M^* is crisper than M.

A4 (Symmetry): This is straightforward by the definition of $H_\alpha^{\text{new}}(M)$ and $\mu_{\overline{M}}(q_i) = 1 - \mu_M(q_i)$. Hence, $H_\alpha^{\text{new}}(M)$ satisfies all properties in the axiomatic definition of fuzzy measure. Therefore, $H_\alpha^{\text{new}}(M)$ is a fuzzy measure of FSs.

Theorem 3.3. *For $M, N \in FSs(X)$, $H_\alpha^{\text{new}}(M \cup N) + H_\alpha^{\text{new}}(M \cap N) = H_\alpha^{\text{new}}(M) + H_\alpha^{\text{new}}(N)$.*

Proof. Let

$$X_1 = \{q \in X \mid \mu_M(q) \ge \mu_N(q)\}, \tag{3.17}$$

and

$$X_2 = \{q \in X \mid \mu_M(q) < \mu_N(q)\}. \tag{3.18}$$

where $\mu_M(q)$ and $\mu_N(q)$ are the membership functions of M and N, respectively.

$$\begin{aligned} \text{If } q \in X_1, \text{ then } \mu_{M\cup N} &= \max\{\mu_M(q), \mu_N(q)\} = \mu_M(q) \\ \text{and } \mu_{M\cap N} &= \min\{\mu_M(q), \mu_N(q)\} = \mu_N(q). \end{aligned}$$

$$\begin{aligned} \text{If } q \in X_2, \text{ then } \mu_{M\cup N} &= \max\{\mu_M(q), \mu_N(q)\} = \mu_N(q) \\ \text{and } \mu_{M\cap N} &= \min\{\mu_M(q), \mu_N(q)\} = \mu_M(q). \end{aligned}$$

Now, consider

$$\begin{aligned} &H_\alpha^{\text{new}}(M \cup N) + H_\alpha^{\text{new}}(M \cap N) \\ &= \frac{1}{k(\alpha - \alpha^{-1})}\left[\sum_{i=1}^{k}\left\{\left(\mu_M(q_i)\right)^{\alpha^{-1}} + \left(1 - \mu_M(q_i)\right)^{\alpha^{-1}} - \left(\mu_M(q_i)\right)^{\alpha} + \left(1 - \mu_M(q_i)\right)^{\alpha}\right\}\right] \\ &\quad + \sum_{i=1}^{k}\left[\left\{\left(\mu_N(q_i)\right)^{\alpha^{-1}} + \left(1 - \mu_N(q_i)\right)^{\alpha^{-1}} - \left(\mu_N(q_i)\right)^{\alpha} + \left(1 - \mu_N(q_i)\right)^{\alpha}\right\}\right]. \end{aligned}$$

On simplifying, we get

$$H_\alpha^{\text{new}}(M \cup N) + H_\alpha^{\text{new}}(M \cap N) = H_\alpha^{\text{new}}(M) + H_\alpha^{\text{new}}(N).$$

4. Numerical Examples

In this section, the performance of proposed fuzzy measure $H_\alpha^{\text{new}}(M)$ will be validated based on the following examples. To illustrate the effectiveness and performance of the proposed measure for FSs, some existing fuzzy measures will be adopted for comparison. Therefore, we first recall some widely used fuzzy measures for FSs.

The fuzzy measure proposed by Yager [34] is shown below:

$$H_{Y_1}(M) = 1 - \frac{d_p(M, M^c)}{n^{\frac{1}{p}}}.$$

The fuzzy measure proposed by Kosko [19] is shown below:

$$H_k(M) = 1 - \frac{d_p(M, M_{\text{near}})}{d_p(M, M_{\text{far}})}.$$

The fuzzy measure proposed by Pal and Pal [25] is shown below:

$$H_{\text{Pal}}(M) = \frac{1}{k}\sum_{i=1}^{k}\Big[\mu_M(q_i)e^{1-\mu_M(q_i)} + \big(1-\mu_M(q_i)\big)e^{\mu_M(q_i)}\Big].$$

The fuzzy measure proposed by Li and Liu [21] is shown below:

$$H_{\text{LL}}(M) = \sum_{i=1}^{k} S\big(cr(\xi_p = q_i)\big).$$

The fuzzy measure proposed by Hwang and Yung [11] is shown below:

$$H_{\text{HY}}(M) = \frac{1}{1-e^{\frac{-1}{2}}}\sum_{i=1}^{k}\bigg[\Big(1-e^{-\mu_{M^c}(q_i)}\Big)I_{[\mu_M(q_i)\geq\frac{1}{2}]} + \Big(1-e^{-\mu_M(q_i)}\Big)I_{[\mu_M(q_i)<\frac{1}{2}]}\bigg].$$

The fuzzy measure proposed by Joshi and Satish [18] shown below:

$$H_\alpha^\beta(M) = \frac{\alpha\times\beta}{k(\alpha-\beta)}\Bigg[\sum_{i=1}^{k}\bigg\{\Big(\mu_M(q_i)^\beta + \big(1-\mu_M(q_i)\big)^\beta\Big)^{\frac{1}{\beta}} - \Big(\mu_M(q_i)^\alpha + \big(1-\mu_M(q_i)\big)^\alpha\Big)^{\frac{1}{\alpha}}\bigg\}\Bigg].$$

Example 1. Consider a FS M_1 of $X = \{3, 4, 5, 6, 7\}$. The FS is defined as:

$$M_1 = \{(3, 0.1), (4, 0.3), (5, 0.4), (6, 0.9), (7, 1)\}.$$

Then the modifier for the fuzzy set

$$M = \big\{q, \big(\mu_M(q)\big) \mid q \in X\big\}$$

in X is given by

$$M^k = \big\{q, \big(\mu_M(q)\big)^k \mid q \in X\big\}. \quad (4.1)$$

Based on the operations, Hwang and Yang [11] and Hung and Yang [13] and in equation (4.1), we have:

$$M_1^{\frac{1}{2}} = \{(3, 0.316), (4, 0.548), (5, 0.632), (6, 0.949), (7, 1)\},$$
$$M_1^2 = \{(3, 0.01), (4, 0.09), (5, 0.16), (6, 0.81), (7, 1)\},$$

$$M_1^3 = \{(3, 0.001), (4, 0.027), (5, 0.064), (6, 0.729), (7, 1)\},$$

and

$$M_1^4 = \{(3, 0), (4, 0.008), (5, 0.026), (6, 0.656), (7, 1)\}.$$

We can regard the FS M_1 as "LARGE" on X by considering the characterization of linguistics variables. Correspondingly, to FSs $M_1^{\frac{1}{2}}$, M_1^2, M_1^3 and M_1^4 may be treated as "More or Less Large", "Very LARGE", "Quite Very LARGE", "Very Very LARGE", respectively. The concept of Shannon's entropy has been utilized for simple weighting calculation method (see Wang and Lee [31], Wu et al. [32]). The larger the value of the information entropy, the smaller the information entropy weight (see Li et al. [22]), then the smaller the different alternatives in this specific attribute and the less information the specific attribute provides and the less important this attribute becomes in decision making process (see Wang and Lee [31]). Intuitively, from $M_1^{\frac{1}{2}}$ to M_1^4, the losss of information hidden in them become less. The entropy conveyed by them incresing. So the following realtion holds (see Hwang and Yang [11], Hung and Yang [13], Joshi and Kumar [18]).

$$H(M_1^{\frac{1}{2}}) > H(M) > H(M_1^2) > H(M_1^3) > H(M_1^4). \tag{4.2}$$

To make a comparison, entropy measures $H_{Y_1}(M_1)$, $H_K(M_1)$, $H_{\text{Pal}}(M_1)$, $H_{\text{LL}}(M_1)$, $H_{\text{HY}}(M_1)$, $H_\alpha^\beta(M_1)$, $H_{\alpha(=2)}^{\text{new}}(M_1)$ are employed to faciliate analysis. In Table 1, we have presented the results obtained based on different measures to faciliate comparative analysis.

Table 1: Fuzziness values with different information measures.

FSs	$H_{Y_1}(M_1)$	$H_K(M_1)$	$H_{\text{Pal}}(M_1)$	$H_{\text{LL}}(M_1)$	$H_{\text{HY}}(M_1)$	$H_\alpha^\beta(M_1)$	$H_{\alpha(=2)}^{\text{new}}(M_1)$
$M_1^{\frac{1}{2}}$	0.397	0.220	1.389	0.810	0.505	0.4672	0.3857
M_1	0.360	0.311	1.331	0.723	0.397	0.4672	0.3442
M_1^2	0.167	0.099	1.202	0.378	0.212	0.2834	0.2349
M_1^3	0.145	0.078	1.151	0.870	0.167	0.2202	0.1795
M_1^4	0.151	0.082	1.136	0.692	0.165	0.1906	0.1531

We can note that FS M will be assigned more entropy than the FS $M_1^{\frac{1}{2}}$ when entropy measures $H_{Y_1}(M_1)$, $H_K(M_1)$ and $H_{\text{LL}}(M_1)$ are applied. The ranking orders obtained based on these measures are listed below.

$$H_{Y_1}(M_1^{\frac{1}{2}}) > H_{Y_1}(M_1) > H_{Y_1}(M_1^2) > H_{Y_1}(M_1^4) > H_{Y_1}(M_1^3),$$

$$H_K(M_1) > H_K(M_1^{\frac{1}{2}}) > H_K(M_1^2) > H_K(M_1^4) > H_K(M_1^3),$$

$$H_{\mathrm{LL}}(M_1^3) > H_{\mathrm{LL}}(M_1^{\frac{1}{2}}) > H_{\mathrm{LL}}(M_1) > H_{\mathrm{LL}}(M_1^4) > H_{\mathrm{LL}}(M_1^2).$$

It is shown that these ranked orders do not satisfy intuitve analysis in equation (4.2), while other entropy measures can induce desirable results. In this example $H_{\mathrm{Pal}}(M_1)$, $H_{\mathrm{HY}}(M_1)$, $H_\alpha^\beta(M_1)$ and $H_{\alpha(=2)}^{\mathrm{new}}(M_1)$ perform well. This illustrates that these entropy measures are not robust enough to distinguish the uncertainty of FSs with linguistic information.

Example 2. Take another FS M_2 defines in X. The FS is defined as :

$$M_2 = \{(3, 0.2), (4, 0.3), (5, 0.4), (6, 0.7), (7, 0.8)\}.$$

We calculate $M_2^{\frac{1}{2}}$, M_2^2, M_2^3 and M_2^4. Now we compare only $H_{\mathrm{Pal}}(M_2)$, $H_{\mathrm{HY}}(M_2)$, $H_\alpha^\beta(M_2)$ and $H_{\alpha(=2)}^{\mathrm{new}}(M_2)$.

Table 2: Fuzziness values with $H_{\mathrm{Pal}}(M_2)$, $H_{\mathrm{HY}}(M_2)$, $H_\alpha^\beta(M_2)$ and $H_\alpha^{\mathrm{new}}(M_2)$. .

Fuzzy sets	$H_{\mathrm{Pal}}(M_2)$	$H_{\mathrm{HY}}(M_2)$	$H_{\alpha=0.3}^{\beta=10}(M_2)$	$H_{\alpha(=2)}^{\mathrm{new}}(M_2)$
$M_2^{\frac{1}{2}}$	1.501	0.653	1.1919	0.5092
M_2	1.513	0.616	1.2073	0.4981
M_2^2	1.386	0.490	1.0206	0.4046
M_2^3	1.094	0.393	0.8418	0.3207
M_2^4	1.241	0.298	0.7006	0.2613

Moreover, the results produced by entropy measures $H_{\mathrm{Pal}}(M_2)$, $H_\alpha^\beta(M_2)$ are also not reasonable, which are shown as the equations below.

$$H_{\mathrm{Pal}}(M_2) > H_{\mathrm{Pal}}(M_2^{\frac{1}{2}}) > H_{\mathrm{Pal}}(M_2^2) > H_{\mathrm{Pal}}(M_2^4) > H_{\mathrm{Pal}}(M_2^3),$$

$$H_\alpha^\beta(M_2) < H_\alpha^\beta(M_2^{\frac{1}{2}}) > H_\alpha^\beta(M_2^2) > H_\alpha^\beta(M_2^3) > H_\alpha^\beta(M_2^4),$$

Therefore, the entropy measures $H_{\mathrm{Pal}}(M_2)$, $H_\alpha^\beta(M_2)$ are not suitable for differetiating the information conveyed by FSs. But $H_{\mathrm{HY}}(M_2)$ and $H_\alpha^{\mathrm{new}}(M_2)$ are also satisfy the ranking order in equation (4.2). The effectiveness of proposed fuzzy measure $H_{\alpha(=2)}^{\mathrm{new}}(M_2)$ and $H_{\mathrm{HY}}(M_2)$ is indicated by this example once again. Hence, the proposed measure consider one parameter which increase the flexibility due to the paramter α whereas H_{HY} does not due to the absense of parameters. Therefore, the proposed measure is encouraging. So the presence of parameter in an information measure makes it flexible from application point of view.

4.1. Sensitive analysis

In this section, a sensitivity analysis has been done to demonstrate the proposed information behaviour. Inclusion of a parameter α effets the reliability or fuzzy information. It provides more malleability to the proposed measure for practical purposes. The one parametric models are more flexible and suitable to use in certain situations. To see the impact of the parameter α in the proposed entropy, different parametric values were implemented, then the same grading results were achieved. Outcomes with various values of parameter α are depicted in Table 3. The ranking is as $H_\alpha^{\text{new}}(M_1^{\frac{1}{2}}) > H_\alpha^{\text{new}}(M_1) > H_\alpha^{\text{new}}(M_1^2) > H_\alpha^{\text{new}}(M_1^3) > H_\alpha^{\text{new}}(M_1^4)$ for any value of α It implies that the change of α has no effect on the ranking sequence and hence our proposed approach considers all linguistic information. Figure 1 portray the sensitivity outcomes for the diverse values of α.

Table 3: Fuzziness values with proposed information measures at different values.

Fuzzy sets	$H^{\text{new}}_{\alpha(=3)}(M_1)$	$H^{\text{new}}_{\alpha(=4)}(M_1)$	$H^{\text{new}}_{\alpha(=5)}(M_1)$	$H^{\text{new}}_{\alpha(=6)}(M_1)$	$H^{\text{new}}_{\alpha(=8)}(M_1)$	$H^{\text{new}}_{\alpha(=10)}(M_1)$
$M_1^{\frac{1}{2}}$	0.3233	0.2732	0.2348	0.2051	0.1631	0.1351
M_1	0.2915	0.2500	0.2181	0.1931	0.1571	0.1323
M_1^2	0.2072	0.1845	0.1662	0.1510	0.1276	0.1103
M_1^3	0.1628	0.1472	0.1338	0.1223	0.1042	0.0906
M_1^4	0.1373	0.1221	0.1091	0.0983	0.0817	0.0699

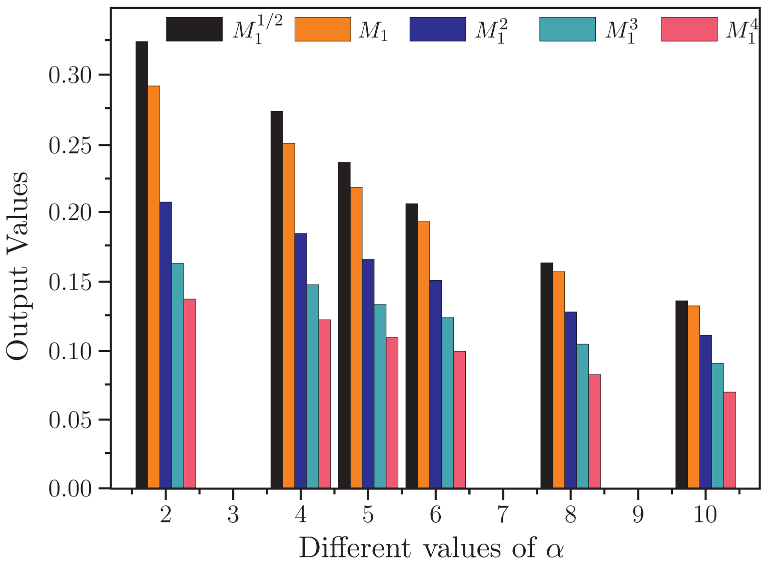

Figure 1: Senstivity analysis of the proposed measure under FSs.

In the next section, we will give the application of the proposed fuzzy measure under bipolar fuzzy theory to find the objective weight. Also, we presented the bipolar fuzzy TOPSIS multi-criteria decision-making model based on entropy weights for the selection of car options with minimum cost and maximum benefits of the intended car. For other terminologies and applications that are not mentioned in this paper, readers may refer to Zhang [37].

5. Bipolar Fuzzy Technique for order preference by similarity to ideal solutions (BF-TOPSIS)

An extension of fuzzy set, called bipolar fuzzy set, was introduced by Zhang [37]. A bipolar fuzzy set is a pair $(\mu_M^+(q), \mu_M^-(q))$, where $\mu_M^+(q) : X \to [0,1]$ and $\mu_M^-(q) : X \to [-1,0]$ are any mappings. $BFSs$ are an extension of fuzzy sets whose membership degree range is $[-1,1]$. In a bipolar fuzzy set, if the membership degree is of an element then we say that the element is irrelevant to the corresponding property, the membership degree $(-1,0]$ of an element implies that the element somewhat satisfies the counter-property and the membership degree $(0,1]$ of an element indicates that the element somewhat satisfies the property. The idea which lies behind such description is connected with the existence of "bipolar information" (e.g., positive information and negative information) about the given set. Positive information represents what is granted to be possible, while negative information represents what is considered to be impossible.

The MCDM problem to be considered can be described such as all alternatives consists of a set denoted by $X = \{q_1.q_2,\ldots,q_k\}$. The set of all considered criteria expressed as $O = \{o_1,o_2,\ldots,o_r\}$. The weight vector of criterias is $w = (w_1,w_2,\ldots,w_k)^T$ with $\sum_{i=1}^k w_i = 1$. Due to limitations of the decision maker knowledge and expertise, a bipolar fuzzy form expresses the evaluation information provided each criteria. The bipolar fuzzy decision matrix given by the decision maker is expressed as :

$$F = \begin{pmatrix} & O_1 & O_2 & \cdots & O_r \\ q_1 & (\mu_{11}^+,\mu_{11}^-) & (\mu_{12}^+,\mu_{12}^-) & \cdots & (\mu_{1r}^+,\mu_{1r}^-) \\ q_2 & (\mu_{21}^+,\mu_{21}^-) & (\mu_{22}^+,\mu_{22}^-) & \cdots & (\mu_{2r}^+,\mu_{2r}^-) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_k & (\mu_{k1}^+,\mu_{k1}^-) & (\mu_{k2}^+,\mu_{k2}^-) & \cdots & (\mu_{kr}^+,\mu_{kr}^-) \end{pmatrix}$$

$w = (w_1,w_2,\ldots,w_k)^T$ such that $0 \le w_j \le 1$ $(j = 1,2,\ldots,k)$ satisfying $\sum_{j=1}^k w_j = 1$. The bipolar fuzzy matrix is calculated as follows : $F = |\xi_{ij}|_{k\times r}$ where $\xi_{ij} = (\xi_{ij}^+,\xi_{ij}^-) = (\mu_{q_i}^+(o_j),\mu_{q_i}^-(o_j))$.

If the attribute weights are completely known, then MCDM problem can be solved by aggregating all bipolar fuzzy information under different attributes and comparing the final bipolar fuzzy values. However in a partial application, the attribute weights are usually partially known or completely unknown. Therefore, the attribute weights must be determined before solving the MCDM problems. The attribute weights can be empirically assigned by decision makers. However, this method is subjective and the partial information on the attribute weights may not be used sufficiently. Therefore, we can propose a new model to determine the attribute weights based on the proposed measure. Generally, we hope the evaluation results of all alternatives under on each attribute are distinguished enough to faciliate our decision making. Therefore, we can set the total information amount as the objective function of optimization. By minimizing the sum of all information amount under all attributes, we can construct the following models.

$$\text{Min} \quad T = \sum_{j=1}^{r} w_j \sum_{i=1}^{k} H_\alpha^{\text{new}}(\mu_{ij}) \tag{5.1}$$

$$\text{s.t.} \quad w \in H, \ \sum_{j=1}^{r} w_j = 1, \quad w_j \geq 0, \ j = 1, 2, \ldots, r,$$

where H is the set of all incomplete information about attribute weights and $H_\alpha^{\text{new}}(\mu_{ij})$ is the information measure calculated by our proposed measure.

When the attribute weights are completely unknown, then we determine the criteria (information entropy) o_j, $1 \leq j \leq r$, using Formula (5.2).

$$E_k(o_j) = \frac{1}{k(\alpha - \alpha^{-1})}\left[\sum_{i=1}^{k}\left\{\left(|\xi_{ij}^-|^{\alpha^{-1}} + (1-\xi_{ij}^+)^{\alpha^{-1}}\right) - \left(|\xi_{ij}^-|^{\alpha} + (1-\xi_{ij}^+)^{\alpha}\right)\right\}\right], \tag{5.2}$$

$1 \leq j \leq r$ and $\alpha > 0 (\neq 1)$.

According to entropy theory, smaller value of entropy across alternatives provides decision makers a useful information. Therefore, criterion should be assigned a bigger weight, otherwise such a criterion will not be given due importance by most of the decision makers. In other words, such a criterion should be evaluated as a very small weight.

In summary, the computational procedure of the decision making method initially introduced by Chen [6] and Hwang and Yoon [10] is listed in the following steps:

1. Calculate the degree of divergence div_j of each criterion o_j using equation (5.3).

$$div_j = 1 - E_k(o_j), \quad 1 \leq j \leq r. \tag{5.3}$$

2. Calculate the entropy weights w_j for each criterion o_j as given in (5.4).

$$w_j = div_j \div \sum_{j=1}^{r} div_j, \quad 1 \leq j \leq r. \tag{5.4}$$

3. Construct the weighted bipolar fuzzy decision matrix $\tilde{M} = [z_{ij}]_{k \times r}$ where, for each $1 \leq i \leq k$, z_{ij} is defined in equation below.

$$z_{ij} = (z_{ij}^+, z_{ij}^-) = w_j(\mu_{ij}^+, \mu_{ij}^-), \quad 1 \leq j \leq r.$$

4. Calculate the Best Solution (q^+) and Worst solution (q^-) using formula (5.5) and (5.6), respectively.

$$q^{+\nu e} = [(\alpha_1^+, \alpha_1^-)\ (\alpha_2^+, \alpha_2^-)\ \cdots\ (\alpha_r^+, \alpha_r^-)]^T, \tag{5.5}$$

$$q^{-\nu e} = [(\beta_1^+, \beta_1^-)\ (\beta_2^+, \beta_2^-)\ \cdots\ (\beta_r^+, \beta_r^-)]^T, \tag{5.6}$$

where $\alpha_j^+ = \inf z_{ij}^+$, $\alpha_j^- = \sup z_{ij}^-$, $\beta_j^+ = \sup z_{ij}^+$, $\beta_j^- = \inf z_{ij}^-$, $1 \le j \le r$.

5. The distance measures of each alternatives q_i from q^+ and q^- using formula (5.7) and (5.8), respectively.

$$d(q_i, q^+) = \sqrt{\sum_{j=1}^{r} \Big[(\mu_{ij}^+ - \alpha_j^+)^2 + (\mu_{ij}^- - \alpha_j^-)^2\Big]}, \tag{5.7}$$

$$d(q_i, q^-) = \sqrt{\sum_{j=1}^{r} \Big[(\mu_{ij}^+ - \beta_j^+)^2 + (\mu_{ij}^- - \beta_j^-)^2\Big]}. \tag{5.8}$$

6. Calculate the relative closeness degree of each alternative q_i using equation (5.9).

$$C(q_i) = \frac{d(q_i, q^{-\nu e})}{d(q_i, q^{+\nu e}) + d(q_i, q^{-\nu e})}, \quad 1 \le i \le k. \tag{5.9}$$

7. Ranking all the alternatives in desending order according to the relative degree of closeness. The alternative away from the q^- and nearest to q^+ will be the best alternative.

Example 3. Assume that we have person X who is confused in choosing a car among five types of cars avaliable in the market. Suppose that he is concentrate on the following features in order to own his car. Price, Color, Elegancy and Safety of a car. Since it is known that every feature effects the cost and benefit of the intended car. So, these cars express the alternatives and the mentioned features represent the criteria in our MCDM problem. Let us denote the concerned cars and criteria by $\{q_1, q_2, q_3, q_4, q_5\}$ and $\{o_1, o_2, o_3, o_4\}$ respectively. Ratings of the alternative, in Table 4 and weights of the criteria are given by a person X in matrices format with bipolar fuzzy and fuzzy values, respectively.

Table 4: Rating of the Alternatives.

Alternatives	Price	Color	Elegancy	Safety
q_1	(0.4,-0.6)	(0.5,-0.6)	(0.8,-0.6)	(0.7,-0.7)
q_2	(0.3,-0.7)	(0.5,-0.6)	(0.6,-0.7)	(0.8,-0.6)
q_3	(0.2,-0.8)	(0.1,-0.8)	(0.9,-0.4)	(0.4,-0.7)
q_4	(0.4,-0.6)	(0.8,-0.1)	(0.8,-0.3)	(0.5,-0.6)
q_5	(0.9,-0.3)	(0.5,-0.7)	(0.4,-0.9)	(0.4,-0.9)

Case 1. Let the partial information avaliable about attributes weight is listed in the following set .

$$H = \{0.10 \le w_1 \le 0.13,\ 0.13 \le w_2 \le 0.19,\ 0.24 \le w_3 \le 0.30,\ 0.40 \le w_4 \le 0.60\}.$$

The overall entropy of each attribute can be calculated by the equations below.

$$K_1 = \sum_{i=1}^{5} \xi_{1i} = \sum_{i=1}^{5} H_\alpha^{\text{new}}(r_{1i}) = 0.102; \qquad K_2 = \sum_{i=1}^{5} \xi_{2i} = \sum_{i=1}^{5} H_\alpha^{\text{new}}(r_{2i}) = 0.055;$$

$$K_3 = \sum_{i=1}^{5} \xi_{3i} = \sum_{i=1}^{5} H_\alpha^{\text{new}}(r_{3i}) = 0.013; \qquad K_4 = \sum_{i=1}^{5} \xi_{4i} = \sum_{i=1}^{5} H_\alpha^{\text{new}}(r_{4i}) = 0.013.$$

The optimal model to determine the attribute weights can be constructed as;

$$\text{Min } T = 0.102w_1 + 0.055w_2 + 0.013w_3 + 0.013w_4$$

$$\text{such that } w \in H, \quad \sum_{j=1}^{4} w_j = 1,\ w_j \ge 0,\ j = 1, 2, 3, 4.$$

Then the weighting vector of the attribute can be obtained as:

$$w = (0.13, 0.19, 0.25, 0.43)^T$$

Table 5: Entropy Weights.

Calculated Values	Price o_1	Color o_2	Elegancy o_3	Safety o_4
$E_n(o_j)$	0.102	0.055	0.013	0.013
w_j	0.13	0.19	0.25	0.43

The weighted fuzzy decision matrix is given in Table 6.

Table 6: Weighted Fuzzy Decision Matrix.

Criteria	Alternatives			
	Price	Color	Elegancy	Safety
o_1	(0.052,-0.078)	(0.095,-0.114)	(0.2,-0.15)	(0.301,-0.301)
o_2	(0.039,-0.091)	(0.095,-0.114)	(0.15,-0.175)	(0.344,-0.258)
o_3	(0.026,-0.104)	(0.019,-0.152)	(0.225,-0.10)	(0.172,-0.301)
o_4	(0.052,-0.078)	(0.152,-0.019)	(0.2,-0.075)	(0.215,-0.258)
o_5	(0.117,-0.039)	(0.095,-0.133)	(0.1,-0.225)	(0.172,-0.387)

The Best and Worst solutions are given in Table 7.

Table 7: Best and Worst Solutions.

α_j^+	0.026	0.019	0.1	0.172
α_j^-	-0.039	-0.019	-0.075	-0.258
β_j^+	0.117	0.152	0.225	0.344
β_j^-	-0.104	-0.152	-0.225	-0.387

The distance measures and ralative closeness degree of each alternative measure are given in Table 8.

Table 8: Distance Measures and Relative Closeness Degree.

Calculated values	q_1	q_2	q_3	q_4	q_5
$d(q_i, q^+)$	1.5044	1.4943	1.4296	1.4170	1.6619
$d(q_i, q^-)$	1.1593	1.1533	1.1847	1.1404	1.3511
$c(q_i)$	0.4352	0.4356	0.4532	0.4459	0.4484

Ranking the alternatives in descending order, we get the following sequence : $C(q_3) \succ C(q_5) \succ C(q_4) \succ C(q_2) \succ C(q_1)$ and $C(q_3)$ is the best available option. Therefore, third car is the most advantageous among the set of cars under study.

Case 2. When there is no infromation for the attribute weights, then the weights can be obtained from equations (5.2), (5.3) and (5.4) and listed in the following Table 9 and 10 .

Table 9: Entropy Weights.

Calculated Values	Price o_1	Color o_2	Elegancy o_3	Safety o_4
$E_n(o_j)$	0.102	0.055	0.013	0.013
div_j	0.898	0.945	0.987	0.987
w_j	0.235	0.248	0.259	0.259

Table 10: Weighted Fuzzy Decision Matrix.

Criteria	Alternatives			
	Price	Color	Elegancy	Safety
q_1	(0.094,-0.141)	(0.124,-0.149)	(0.207,-0.155)	(0.181,-0.181)
q_2	(0.071,-0.165)	(0.124,-0.149)	(0.155,-0.181)	(0.207,-0.155)
q_3	(0.047,-0.188)	(0.025,-0.198)	(0.233,-0.104)	(0.104,-0.181)
q_4	(0.094,-0.141)	(0.198,-0.248)	(0.207,-0.078)	(0.130,-0.155)
q_5	(0.212,-0.071)	(0.124,-0.174)	(0.104,-0.233)	(0.104,-0.233)

The Best and Worst solutions are given in Table 11.

Table 11: Best and Worst Solutions.

α_j^+	0.047	0.025	0.104	0.104
α_j^-	-0.071	-0.025	-0.078	-0.015
β_j^+	0.212	0.198	0.233	0.207
β_j^-	-0.188	-0.198	-0.233	-0.233

The distance measures and relative closeness degree of each alternative measure are given in Table 12.

Table 12: Distance Measures and Relative Closeness Degree.

Calculated values	q_1	q_2	q_3	q_4	q_5
$d(q_i, q^+)$	1.5400	1.5275	1.4311	1.4324	1.6936
$d(q_i, q^-)$	1.1723	1.1666	1.1331	1.1219	1.3355
$c(q_i)$	0.4322	0.4330	0.4419	0.4392	0.4409

All alternatives can be ranked into the following order: $C(q_3) \succ C(q_5) \succ C(q_4) \succ C(q_2) \succ C(q_1)$, according to relative closeness degree, we conclude that third car is the most advantageous among the set of cars under study.

For comparative analysis, we can also solve this multi criteria decision making (MCDM) problem by applying Alghandi et al. [3] method. The ranked order of five alternatives is $C(q_3) \succ C(q_5) \succ C(q_4) \succ C(q_2) \succ C(q_1)$. We can see that both of our proposed method based on (5.2) and the method proposed by Alghandi et al. [3], we can take $C(q_3)$ as the best choice for choosing a car. Even though the all order are different. This difference has no effect on choosing the best alternative for cars. Actually, the solution of an MCDM problem only concerns the best alternative. The order of other alternatives is beyond the ultimate goal of an MCDM problem. This example demonstartes that the proposed methods for solving MCDM problems are competent to getting reasonable results. Compared with Alghandi et al. [3] method, our proposed optimal model is easier, which will reduce the computation burden. The fuzzy decision making method with the entropy weights is more effective and practical for dealing with the partially known and unknown information about criteria weights. Thus, the priority of the new information measure is also verified.

5.1. Managerial implications

Multi-criteria decision-making is a procedure to make an ideal decision that has the highest level of achievement from a set of alternatives that are portrayed with respect to different conflicting criteria. TOPSIS method is the most favorable and effective method to resolve MCDM problems. To deal with uncertainty and incomplete information, fuzziness, intuitionistic fuzziness and neutrosophic sets have been used successfully in TOPSIS methods for solving MCDM problems. However, in many cases, the given information is bipolar in nature. Recently, a bipolar fuzzy TOPSIS method for the reasonable selection of objects was discussed by Alghandi et al. [3]. But, in this method, the weights are chosen arbitrarily, which can be changed according to the choice of decision-makers. The chosen weights may be irrelevant for the given information, which can effect the results of decision-making. Therefore, it is important to calculate weights (completely known and unknown) as indicated by the given information. In our method, we have talked about the procedure for calculating entropy weights from given bipolar fuzzy information. It gives more reasonable decisions as compared to the previous methods discussed in the literature. Therefore, current study have focused on evaluating car services to guide for the consumers to select the best car services. The ranking results obtained is more reliable and accurate since it avoids the situation of having the same similarity index to both positive and negative ideal solutions. Thus, this evaluation model can be applied in other scenarios which have same characteristics as compared to those of car service industry and it can be used widely in the area of bargaining process which is usually uncertain and complex. There are many categories of cars including SUVs, Sedans, Crossovers, MPVs etc. Due to this diversity, it is difficult for consumers to opt a particular car. Therefore, our study also provides theoretical and practical guidance for the consumers that are intending to choose the best offer.

6. Conclusions

In this paper, we have successfully introduced a new information measure involving one parameter based on Havrda-Charvat-Tsallis entropy and their necessary properties are verified. The proposed information measure has been compared with existing entropies. Some numerical examples based on linguistic terms have been offered to show the effectiveness and applicability of the proposed information measure. Further, we developed a new framework for tackling bipolar fuzzy information by combining the notion of bipolar fuzzy sets and TOPSIS method. It ranks all alternatives in decreasing order. The best alternative is clearly identified. The proposed method based on evaluating the distances of each alternative to bipolar fuzzy Best and Worst solutions. Also, we have displayed the methodology of the bipolar fuzzy TOPSIS method based on entropy weights. For illustration, we have applied this method to a real-life problems. The proposed entropy and MCDM method can further be applied to the concept of the parametric directed divergence measure, similarity and dissimilarity measures for fuzzy sets, intuitionistic fuzzy sets, pythagorean fuzzy sets, coding theory, interval valued intuitionistic fuzzy sets, picture fuzzy sets etc.

Acknowledgements

The authors are thankful to the anonymous reviewers for their valuable suggestions and comments to improve this manuscript.

References

[1] Aczel, J. and Daroczy, Z. (1975). On measures of information and their characterization, Academic Press, New York.

[2] Arimoto, S. (1971). *Information-theoretical considerations on estimations problems*, Information Control, Vol.19, 181-191.

[3] Alghamdi, M. A., Alshehri, N. O. and Akram, M. (2018). *Multi-criteria decision making methods in bipolar fuzzy environment*, International Journal of Fuzzy Systems, Vol.20, 5-6.

[4] Boekee, D. E. and Vander Lubbe, J. C. A. (1980). *The R-norm information measure*, Information Control, Vol.45, 136-155.

[5] Bhandari, D. and Pal, N. R. (1993). *Some new information measures for fuzzy sets*, Information Sciences , Vol.67, No.3, 204-228.

[6] Chen, C. T. (2000). *Extension of the TOPSIS for group decision-making under fuzzy environment*, Fuzzy Sets System, Vol.1, No.114, 1-9.

[7] De Luca, A. and Termini, S. (1972). *A definition of a non-probabilistic entropy in the setting of fuzzy sets theory*, Information Control, Vol.20, 301-312.

[8] Havrda, J. H. and Charvat, F. (1967). *Quantification method classification process: concept of structral - entropy*, Kybernetika, Vol.3, 30-35.

[9] Hooda, D.S. (2004). *On generalized measures of fuzzy entropy*, Mathematica Slovaca, Vol.54, 315-325.

[10] Hwang, C. L. and Yoon, K. (1981). Multiple attribute decision-making-methods and applications, In A State of the Art Survey; Springer, New York, USA.

[11] Hwang, C. H. and Yang, M. S. (2008). *On entropy of fuzzy sets*, International Journal of Uncertainty and Fuzziness Knowledge-Based System, Vol.16, 519-527.

[12] Higashi, M. and Klir, G. J. (1982). *On measures of fuzziness and fuzzy complements*, International Journal of General Systems, Vol.8, 169-180.

[13] Hung, W. L. and Yang, M. S. (2006). *Fuzzy entropy on intuitionistic fuzzy sets*, International Journal of Intelligient Systems, Vol.21, 443-451.

[14] Kumar, S., Choudhari, A., Ram, G. and Gupta, V. (2013). *On useful relative information measures of order α and type β*, International Journal of Information and Management Sciences, Vol.24, 95-106.

[15] Joshi, R. and Kumar, S. (2016). *(R, S)-norm information measure and a relation between coding and questionnaire theory*, Open Systems and Information Dynamics, Vol.23, No.3, 1-12.

[16] Joshi, R. and Kumar, S. (2016). *A new approach in multiple attribute decision making using R-norm entropy and Hamming distance measure*, International Journal of Information and Management Sciences, Vol.27, No.3, 253-268.

[17] Joshi, R. and Kumar, S. (2019). *A new approach in multiple attribute decision making using exponential hesitant fuzzy entropy*, International Journal of Information and Management Sciences, Vol.30, No.4, 305-322.

[18] Joshi, R. and Kumar, S. (2018). *An (R, S)-norm fuzzy information measure with its applications in multiple-attribute decision-making*, Computational and Applied Mathematics, Vol.37, No.3, 2943-2964.

[19] Kosko, B. (1986). *Fuzzy entropy and conditioning*, Information Sciences, Vol.40, No.2, 165-174.

[20] Kumar, S. and Kumar, A. (2012). *Some coding theorem on generalized Havrda-Charvat-Tsallis entropy*, Tamking Journal of Mathematics. Vol.43, No.1, 437-444.

[21] Li, P. and Liu, B. (2008). *Entropy of credibility distributions for fuzzy variables*, IEEE Transactions on Fuzzy Systems, Vol.16, No.1, 123-129.

[22] Li, X., Wang, K., Liu, L., Xin, J., Yang, H. and Gao, C. (2011). *Application of the entropy weight and TOPSIS method in safety evaluation of coal mines*, Procedia Engineering, Vol.26, 2085-91.

[23] Mishra, A. R. and Rani, P. (2018). *Biparametric information measures-based TODIM technique for interval-valued intuitionistic fuzzy environment*, Arbian Journal of Science and Engineering, Vol.43, No.6, 3291-3309.

[24] Pal, N. R. and Pal, S. K., (1989). *Object background segmentation using new definition of entropy*, Proc. Inst. Electron 671 Eng, Vol.136, 284-295.

[25] Pal, N. R. and Pal S. R., (1992). *Higher order fuzzy entropy and hybrid entropy of a set*, Information Sciences, Vol.61, No.3, 211-231.

[26] Renyi, A. (1961). *On measures of entropy and information*, In: Proceedings of the 4th Barkley symposium on Mathemtaical statistics and probability, University of California Press, Vol.1, 547.

[27] Shannon, C. E. (1948). *A mathematical theory of communication*, The Bell System Technical Journal, Vol.27, No.3, 378-423.

[28] Sharma, B. and Taneja, I. (1975). *Entropies of type* (α, β) *and other generalized measures of information theory*, Metrika, Vol.22, 202-215.

[29] Tsalli, C. (1988). *Possible generalization of Boltzman-Gibbs statistics*, Journal of Statistical Physics, Vol.52, No.1-2, 480-487.

[30] Verma, R. and Sharma, B. D. (2011). *On generalized exponential fuzzy entropy*, International Journal of Mathematical and Computational Sciences, Vol.5, No.12, 1402-1405.

[31] Wang, T. C. and Lee, H. D. (2009). *Developing a fuzzy TOPSIS approach based on subjective weights and objective weights*, Expert Systems with Applications, Vol.36, No.5, 8980-5.

[32] Wu, S., Fu, Y., Shen, H. and Liu, F. (2018). *Using ranked weights and Shannon entropy to modify regional sustainable society index*, Sustainable Cities and Society, Vol.41, 443-448.

[33] Xiong, S. H., Wu, S., Chen, Z. S. and Li, Y. L. (2017). *Generalized intuitionistic fuzzy entropy and its application in weight dteremination*, Control and Decision, Vol.32, No.5, 845-854.

[34] Yager, R. R. (1979). *On the measure of fuzzi ness and negation. Part* 1 *: membership in the unit interval*, International Journal of General Systems, Vol.5, No.4, 221-229.

[35] Zadeh, L. A. (1965). *Fuzzy sets*, Information and Control, Vol.8, 338-353.

[36] Zadeh, L. A. (1968). *Probability measures of fuzzy events*, Journal of Mathematical Analysis and Applications, Vol.23, 421-427.

[37] Zhang, W. R. (1994). *Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis*, In: Proceedings of IEEE Conference Fuzzy Information Processing Society Biannual Conference, 305-309.

Department of Mathematics, Maharishi Markandeshwar University, Mullana-Ambala 133207, India.

E-mail: vikasarya1988@gmail.com

Major area(s): Information coding theory, measure theory, fuzzy information measure, decision making problems, fuzzy entropy.

Department of Mathematics, Maharishi Markandeshwar University, Mullana-Ambala 133207, India.

E-mail: drsatish74@rediffmail.com

Major area(s): Information coding theory, measure theory, topology, fuzzy information measure, decision making problems, fuzzy entropy, pattern recognition.

(Received March 2019; accepted February 2020)