當　代　會　計
第二十六卷第一期
民國一一四年五月
頁 1-29

Journal of Contemporary Accounting
Vol. 26 No. 1, May 2025
PP.1-29

財務會計及審計研究之基石－盈餘品質及審計品質之檢視

王泰昌* 陳良軒**

摘要：盈餘品質及審計品質為財務會計及審計領域之基本且重要研究議題，本文檢視財務會計及審計領域學者過去發表之文獻，評論過去文獻之缺失，並針對缺失提出本文之意見及想法。本文將區分為三部分。第一部分，本文探討盈餘之定義及盈餘品質，提出 Dechow, Ge, and Schrand (2010)一文及相關文獻中之缺失，並針對缺失提出意見及想法。第二部分，本文探討審計之定義及審計品質，提出 DeFond and Zhang (2014)一文中之缺失，並針對缺失提出意見及想法。第三部分總結全文。

關鍵詞：盈餘品質、盈餘持續性、審計品質、財務報導品質

* 國立臺灣大學會計學系教授、國立臺灣大學計量理論與應用研究中心（通訊作者，E-mail: tcwang@ntu.edu.tw）
** 國立臺灣大學會計學系博士候選人（E-mail: d10722003@ntu.edu.tw）
作者感謝教育部 Grant no. 112L900201 的經費補助。

113 年 06 月收稿
113 年 09 月接受
DOI: 10.6675/JCA.202505_26(1).0001

Journal of Contemporary Accounting
Vol. 26 No. 1, May 2025
PP.1-29

The Foundation of Financial Accounting and Auditing Research: Revisiting Earnings Quality and Audit Quality

Taychang Wang*　Liang-Shiuan Chen**

Abstract: Earnings quality and audit quality are fundamental and crucial research topics in the fields of financial accounting and auditing. We revisit the literature published by scholars in the fields of financial accounting and auditing, critique the shortcomings of the literature, and present our opinions and thoughts on these shortcomings. The paper is divided into three parts. In the first part, we discuss the definitions of earnings and earnings quality, identify the shortcomings in Dechow, Ge, and Schrand (2010) and related papers, and present our opinions and thoughts on these shortcomings. In the second part, we discuss the definition of auditing and audit quality, identify the shortcomings in DeFond and Zhang (2014), and present our opinions and thoughts on these shortcomings. We then provide concluding remarks in the third part.

Keywords: earnings quality, earnings persistence, audit quality, financial reporting quality

* Professor, Department of Accounting, National Taiwan University, Research Fellow, Center for Research in Econometric Theory and Applications, National Taiwan University (Corresponding author, E-mail: tcwang@ntu.edu.tw)
** Ph.D. Candidate, Department of Accounting, National Taiwan University (E-mail: d10722003@ntu.edu.tw)
This work was financially supported by the Ministry of Education (MOE) Grant no. 112L900201.
Submitted June 2024
Accepted September 2024
DOI: 10.6675/JCA.202505_26(1).0001

I. Definitions of Earnings and Earnings Quality

In the field of financial accounting, both managers and investors rely heavily on earnings figures in the financial statements as measures of firm performance. Therefore, defining and measuring earnings have become core issues in the field of financial accounting. In the field of financial accounting, earnings are generally defined as the earnings calculated in accordance with generally accepted accounting principles (GAAP). In this paper, we label this type of earnings as "accounting earnings." In addition to accounting earnings, common definitions and measurement methods for earnings include "economic earnings" and "perpetual earnings." These three types of definitions and measurement methods of earnings will be illustrated in the following example.

Assume that a firm currently holds \$500,000 in cash and uses the \$500,000 to purchase a building. This building will be leased to others for use, generating an annual rental income of \$50,000. Also assume an estimated useful life of the building of 50 years with no salvage value. Straight-line depreciation is used. One year later, a real estate appraiser evaluates the fair value of the building at \$520,000.

Using the three aforementioned definitions and measurement methods, the earnings for the current period are calculated as follows:

- **Accounting earnings:** Accounting earnings are computed based on the rental income received during the period (\$50,000) minus the depreciation expense recorded (\$10,000). Therefore, the accounting earnings are \$50,000 − \$10,000 = \$40,000.
- **Economic earnings:** Economic earnings are determined by considering the rental income received during the period (\$50,000) plus the increase in the value of the building one year later (\$20,000). Thus, the economic earnings amount to \$50,000 + \$20,000 = \$70,000.
- **Perpetual earnings:** Perpetual earnings represent the rental income that can be continuously received in each period in the future. In this case, the perpetual earnings are \$50,000.

From the calculations in the example above, it is evident that the earnings figures derived from the three definitions vary. First, as mentioned earlier, accounting earnings refer to earnings calculated in accordance with generally accepted accounting principles (GAAP). Therefore, accounting earnings reflect the accounting standards applicable to the firm. According to GAAP, firms are required to record depreciation expense based on the accrual basis. Hence, in this example, the accounting earnings of the firm, in addition to rental income, also deduct the depreciation expense. However, the depreciation expense is affected by the firm's estimation of the useful life of the building and the method of recording depreciation. In other words, accounting earnings may be subject to manipulation by the firm. Furthermore, accounting earnings may not necessarily reflect the true value of the firm. In this example, while the value of the building increases one year later, it is not reflected in the accounting earnings at all.

Economic earnings primarily measure the cash flows generated during the current period and the changes in the present value of future cash flows. In this example, the current cash flows are the rental income of $50,000. As for the changes in the present value of future cash flows, they are reflected in the increase in the value of the building ($20,000). Therefore, the sum of these two amounts ($70,000) represents the economic earnings for the current period. Unlike accounting earnings, economic earnings truly reflect the changes in the firm's value and are closer to the concept of "comprehensive income" in generally accepted accounting principles. However, the value of the building may change in each single period, making economic earnings less predictive of future earnings.

Perpetual earnings primarily measure the expected earnings that a firm can consistently earn in each period in the future and can be thought of as the average of long-term economic earnings. In this example, assuming that the average of future changes in the value of the building is zero, the perpetual earnings of the firm would be the rental income of $50,000. Calculating perpetual earnings generally aims to determine the current value of the firm. By dividing the calculated perpetual earnings by an appropriate discount rate, the current value of the firm can be obtained. However, in general, firms cannot predict future changes in the value of the building over multiple periods. Therefore, perpetual earnings are often hard to calculate.

Figure 1 compares economic earnings and perpetual earnings. From Figure 1, it can be understood that economic earnings reflect the changes in firm value in each period. If the value of the building fluctuates significantly in each period, it will result in significant fluctuations in economic earnings in each period. On the other hand, perpetual earnings represent the average of long-term economic earnings.

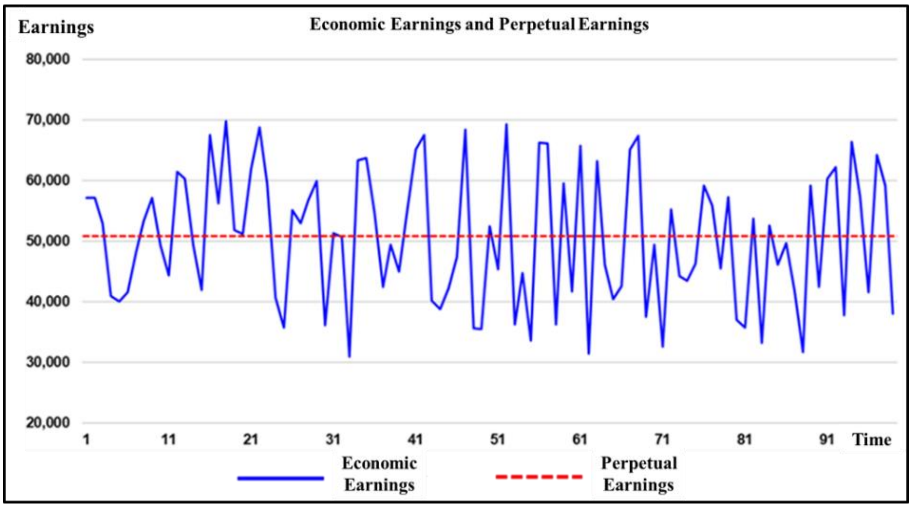

Figure 1: Economic Earnings and Perpetual Earnings

After discussing the definitions and measurement methods of the three types of earnings, we proceed to comment on the work of Dechow, Ge, and Schrand (2010) and related papers, highlight some shortcomings, and present our opinions and thoughts on these shortcomings. Dechow et al. (2010) review a lot of literature related to earnings quality over the past several years and discuss the measures of earnings quality. First, Dechow et al. (2010) argue that higher earnings quality should more faithfully reflect the features of the firm's fundamental earnings process and be more relevant to specific decision-makers when making specific decisions. They particularly emphasize "specific decision-makers" and "specific decisions" as different decision-makers focus on different earnings characteristics when making decisions. For example, banks are more concerned about the firm's future solvency when deciding whether to approve a loan and would prefer stable earnings rather than significant fluctuations. In contrast, investors willing to bear higher risks may be more accepting of significant earnings fluctuations.

In the following, we compare the pre-publication and the published versions of Dechow et al. (2010) and comment on specific contexts. In the pre-publication version, Dechow et al. (2010) define the reported earnings of a firm and the fundamental earnings process (X) as follows:

Reported Earnings = Function of (X) + *error induced by accounting system* (e) (1)

In equation (1), the reported earnings of a firm are considered to be a function of the firm's fundamental earnings process, X, plus the error term generated by the firm's accounting system. The term "function" here represents the transformation of the fundamental earnings process into the reported earnings by the firm's accounting system. Additionally, in the pre-publication version, Dechow et al. (2010) define the fundamental earnings process of a firm as

"*the output of the firm's production function or business model and can be thought of as the expected cash flows generated during the period that could be annuitized to obtain the fundamental value of the firm, alternatively referred to as perpetual earnings.*"

Dechow et al. (2010) argue that the fundamental earnings process of a firm represents the output of the firm's production function or business model and can be conceptualized as the expected cash flows over a period, which can be used to calculate the firm's value, i.e., the concept of **perpetual earnings**. As mentioned earlier, after dividing perpetual earnings by an appropriate discount rate, the current value of the firm can be calculated. However, Dechow et al. (2010) use the term **"annuitized"** to describe this discounting process, which is exactly in contrast to the concept of discounting. Discounting refers to calculating the present value of future cash flows by dividing them by an appropriate discount rate, while annuitizing refers to converting a known present value into future cash flows.

Furthermore, we argue that the formulation proposed by Dechow et al. (2010) lacks rigor. The formulation in equation (1) does not take the heterogeneity of different firms and different time points into account. The fundamental earnings processes of different firms at different time points are different, and the functional forms of these processes also differ due to different firms and different time points. Therefore, we suggest that equation (1) should be expressed more rigorously as follows:

$$Reported\ Earnings_{i,t} = Function_{i,t}(X_{i,t}) + error\ induced\ by\ accounting\ system\ (e_{i,t}) \qquad (2)$$

In the published version, Dechow et al. (2010) still emphasize that earnings quality depends on specific decision-makers and specific decisions. However, the definition of reported earnings alters. In the published version, Dechow et al. (2010) define the reported earnings of a firm as a function of firm performance of the firm and express it as the following equation:

$$Reported\ Earnings = f(X) \qquad (3)$$

X in equation (3) represents the firm performance. Additionally, Dechow et al. (2010) define firm performance using the concept of generally accepted accounting principles. If the firm's lifespan is only one year, then the firm performance for the current period is defined as

"*the cash flows generated during the period plus the change in the liquidation value of net assets*."

If the firm's lifespan is multiple years, then the firm performance for the current period is the sum of the following three components:

"(*i*) *cash flows generated during the current period,* (*ii*) *the present value of cash flows that will be generated in future periods that are a result of actions taken in the current period, and* (*iii*) *the present value of the change in the liquidation value of net assets that are a result of actions taken in the current period.*"

At first glance, this definition seems clear and reasonable. However, upon closer examination, it can be noticed that the second component is actually included within the third component. In other words, this definition can be simplified to include only the first and the third components, with the second component being redundant and can be removed.

It is also worth noting that under this definition, earnings correspond to the **economic earnings** mentioned earlier, i.e., the sum of the cash flows generated during the current period and the changes in the present value of future cash flows. This definition is completely different from the pre-publication version, which defines earnings as perpetual earnings. The published version changes the definition from perpetual earnings to economic earnings. We suggest that using economic earnings as the definition is preferable because perpetual earnings are often difficult to calculate, while economic earnings can reflect changes in the firm's value in the current period. Economic earnings are also easier to calculate compared to perpetual earnings.

Dechow et al. (2010) revise the definition of earnings to economic earnings in the published version and further reference the earnings described by Penman and Sougiannis (1998) in the article. However, the description of earnings stated by Penman and Sougiannis (1998) is inconsistent with the definition of economic earnings and is more akin to the definition of **accounting earnings**. Therefore, we argue that Dechow et al. (2010) do not provide a clear and consistent discussion on how earnings should be defined and measured.

Even though Dechow et al. (2010) define the earnings as economic earnings in the published version, they assert that how to define earnings is still open to debate as follows:

"*Should earnings measure changes in fair value* (*current or exit prices*) *of an enterprise, or should earnings measure "sustainable" cash flows, such that it can be annuitized to reflect value?*"

The statement is quite confusing for readers. Dechow et al. (2010) have already defined earnings as economic earnings. However, it seems that, from the statement above, they are not confident with their definition. Besides, the literature they cite further confuses readers. For example, they cite the description of earnings stated by Penman and Sougiannis (1998), but the description is inconsistent with what Dechow et al. (2010) discuss. In sum, we argue that it is hard to understand the contexts as Dechow et al. (2010) do not provide a clear discussion.

Furthermore, the formulation of equation (3) mentioned earlier lacks rigor. In addition to overlooking the heterogeneity among firms and time points, equation (3) directly removes the error term compared to equation (1) in the pre-publication version. We argue that removing the error term generated by the accounting system is incorrect, and a more accurate and rigorous formulation should follow the expression of equation (2).

We continue to comment on the discussion in Dechow et al. (2010) regarding the proxy variables for earnings quality and present our views. We discuss the statistical properties of earnings, which involve calculating parameters related to earnings quality using statistical methods, such as earnings persistence. Dechow et al. (2010) point out that prior literature in the field of financial accounting often measures earnings quality using earnings persistence. It is generally accepted that higher earnings persistence indicates greater sustainability of earnings, further implying higher earnings quality. The measurement of earnings persistence involves using current period earnings as the dependent variable and prior period earnings as the independent variable in regression analysis as follows:

$$X_{j,t} = \varphi_{0,j} + \varphi_{1,j} X_{j,t-1} + \upsilon_{j,t}, \tag{4}$$

where:

$X_{j,t}$ = the earnings per share of firm j at time t, and

$X_{j,t-1}$ = the earnings per share of firm j at time t-1

In equation (4), $\varphi_{0,j}$ represents the intercept term, $\varphi_{1,j}$ represents the slope term, and $\upsilon_{j,t}$ represents the error term. Here, $\varphi_{1,j}$ is the parameter used to measure earnings persistence. Prior literature suggests that when $\varphi_{1,j}$ approaches 1, it indicates higher earnings persistence, implying higher earnings quality. Conversely, when $\varphi_{1,j}$ approaches 0, it indicates lower earnings persistence, implying lower earnings quality. For example, Francis, Olsson, and Schipper (2008) state:

"*The resulting estimate of* $\varphi_{1,j}$ *captures firm j's persistence of earnings. Values of* $\varphi_{1,j}$ *close to one imply highly persistent* (*i.e., high quality*) *earnings, while values of* $\varphi_{1,j}$ *close to zero imply highly transitory* (*i.e., low quality*) *earnings.*"

However, we argue that this is an incorrect statement. Through simulations using statistical software, we find that the direction of this assertion is in fact opposite. When we set $\varphi_{1,j}$ in equation (4) to be 1 and $\varphi_{0,j}$ to be 0 and simulate 1,000 periods, we obtain the result shown in Figure 2. Conversely, if we set $\varphi_{1,j}$ to be 0 and $\varphi_{0,j}$ to be 0 in equation (4) and simulate 1,000 periods, we obtain the result shown in Figure 3.

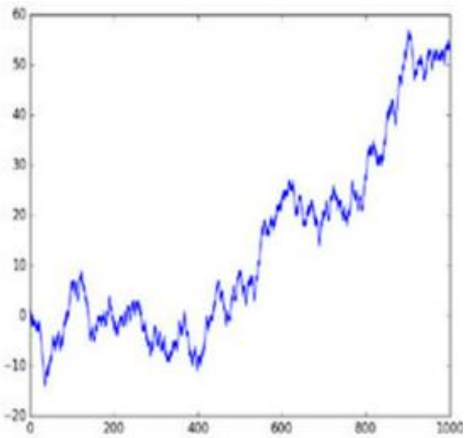

Figure 2: $\varphi_{1,j} = 1$; $\varphi_{0,j} = 0$

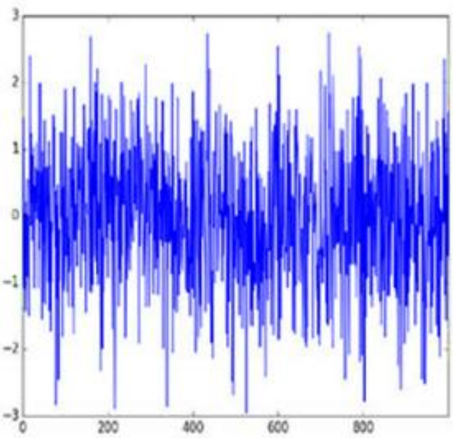

Figure 3: $\varphi_{1,j} = 0$; $\varphi_{0,j} = 0$

From the simulation results shown in Figures 2 and 3, it is observed that when $\boldsymbol{\varphi_{1,j}}$ = 1, the trend of earnings resembles a random walk pattern, with significant differences in earnings among periods. Conversely, when $\boldsymbol{\varphi_{1,j}}$ = 0, the trend of earnings resembles a white noise pattern, with earnings fluctuating around 0 in each period. Comparing the two figures, it is clear that earnings persistence is higher in Figure 3 and lower in Figure 2, which is contrary to what is stated in the prior literature. Notably, the scale in Figure 2 ranges from -20 to +60, while in Figure 3, the scale ranges from -3 to +3. If we were to overlay the two blue lines in one figure, the line in Figure 3 would be much more stable than the line in Figure 2.

Furthermore, whether higher earnings persistence necessarily implies higher earnings quality is also worth exploring. As mentioned earlier, if decision-makers intend to use earnings to calculate the firm's current value, then earnings in Figure 3 (i.e., when $\boldsymbol{\varphi_{1,j}}$ = 0) are more accurately reflective of the firm's value. When $\boldsymbol{\varphi_{1,j}}$ = 0, earnings in each period are closer to each other, resembling the concept of perpetual earnings, which can more accurately determine the firm's value. Thus, higher earnings persistence can represent higher earnings quality. However, if the trend of perpetual earnings does not match the actual stock price trend of the firm, then a higher degree of earnings persistence does not necessarily imply higher earnings quality. In other words, if the actual stock price trend of the firm aligns more with a random walk, then the earnings quality is higher in Figure 2. From this discussion, we argue in this paper that earnings persistence is not necessarily positively correlated with earnings quality. Therefore, this paper suggests revising the statement made by Francis et al. (2008) as follows:

"*The resulting estimate of $\varphi_{1,j}$ captures firm j's persistence of earnings. Values of $\varphi_{1,j}$ close to **one** imply highly **transitory** earnings, while values of $\varphi_{1,j}$ close to **zero** imply highly **persistent** earnings.*"

Both Francis et al. (2008) and Dechow et al. (2010) mention earnings persistence when discussing earnings quality, and both of them state that when $\boldsymbol{\varphi_{1,j}}$ approaches 1, it indicates higher earnings persistence, while when $\boldsymbol{\varphi_{1,j}}$ approaches 0, it indicates lower earnings persistence. However, this paper finds through simulations that the assertions made in the prior literature are incorrect. Additionally, we also believe that linking higher earnings persistence with higher earnings quality is inappropriate.

Another serious mistake of Dechow et al. (2010) centers around several equations related to predicting earnings: "(1a) $Earnings_{t+1} = \alpha + \beta Earnings_t + \varepsilon_t$, (1b) $Earnings_{t+1} = \alpha + \beta_1 CF_t + \beta_2 Accruals_t + \varepsilon_t$, and (1c) $Earnings_{t+1} = \alpha + \delta_1 Earnings_t + \delta_2 Financial\ statements\ components_t + \delta_3 Other\ information_t + \varepsilon_t$." (Dechow et al. 2010, p.352) These equations are incorrect. The time period of the error term in each equation should be t+1 instead of t. Lack of knowledge about time series analysis may be the cause of the flaw, and it seems that the errors do not occur because of typos.

To better understand the concepts of random walk and white noise, we further use the model in Ball, Kothari, and Nikolaev (2013) as another example and comment on some contexts in this work. Ball et al. (2013) build a model to study the coefficient of asymmetric timeliness across firms. The model is as follows:

$$R_t = x_t + y_t + g_t, \tag{5}$$

$$I_t = x_t + w_t y_t + (1 - w_{t-1}) y_{t-1} + g_{t-1} + \varepsilon_t - \varepsilon_{t-1}, \tag{6}$$

where:

R_t = total unexpected security return,

I_t = accounting earnings,

x_t = portion of the total unexpected return that invariably is contemporaneously captured in I_t,

y_t = portion of the total unexpected return that is not contemporaneously captured in I_t unless required by conservative accounting,

g_t = portion of the total unexpected return that never is contemporaneously captured in I_t but always is incorporated with a lag,

w_t = an indicator variable that takes the value of 1 when conservative accounting rules lead to recognition of y in the current period, and

ε_t = noise in accounting earnings that reverses in the next period.

Equations (5) and (6) specify the relationship between stock return and accounting earnings. In the equations, it is observed that some components are not contemporaneously captured by accounting earnings but are immediately reflected in stock return. The noise ε_t arises from the accruals recorded in the accounting earnings.

We argue that the two equations are problematic. From equation (6), it can be seen that the pattern of the accounting earnings is akin to a white noise pattern. However, the stock price typically resembles a random walk pattern. The problem arises from the inconsistent patterns between the accounting earnings and the stock price. Typically, the stock price is obtained by discounting future cash flows by an appropriate discount rate, and it is impossible to derive the stock price resembling a random walk pattern from the accounting earnings akin to a white noise pattern.

Besides, equation (5) directly ignores the error term, which makes the model unrealistic. The factors affecting the stock return are complicated, which include not only the components captured by the accounting system. For example, the sentiment of investors or irrational behaviors of investors, to name a few. We argue that the model in Ball et al. (2013) is problematic and the two equations are essentially unrealistic.

From an accounting standpoint, there are additional shortcomings related to the model. Ball et al. (2013) provide the following correlation relationships among x_t, y_t, and g_t:

$$\text{corr}(x_t, y_t) > 0, \text{corr}(x_t, g_t) > 0, \text{ and } \text{corr}(x_t, g_t) > 0 \quad (7)$$

$$\text{cov}(x_t, y_t \mid R_t > 0) = \text{cov}(x_t, y_t \mid R_t < 0), \text{ and } \text{var}(x_t \mid R_t > 0) = \text{var}(x_t \mid R_t < 0) \quad (8)$$

According to the definitions of the variables, if the rules of conservative accounting are applied, then y_t will be recognized contemporaneously in the accounting earnings. In other words, conceptually, y_t refers to the item which will negatively impact the accounting earnings, such as impairment losses. It is implied that y_t is a strictly negative variable. For any realization of y_t, y_t will always be negative. If the correlation relationships among x_t, y_t, and g_t are all positive, it implies that all the three components are negatively affecting the accounting earnings, which further implies that the accounting earnings are negative. Generally, this can not be the case in reality. Hence, equation (7) is problematic.

We continue to show that equation (8), which contradicts equations (5) and (7), is also questionable. Given that $R_t = x_t + y_t + g_t$, let us discuss the cases where $R_t > 0$ and $R_t < 0$, respectively. When $R_t > 0$, since $y_t < 0$, for R_t to be positive, $x_t + g_t$ must be sufficiently positive to counteract the negativity of y_t. When $R_t < 0$, again, since $y_t < 0$, $x_t + g_t$ must be insufficiently positive or negative, leading R_t to be negative. The condition cov (x_t, $y_t \mid R_t > 0$) = cov (x_t, $y_t \mid R_t < 0$) implies that the relationship between x_t and y_t does not change no matter R_t is positive or negative. For $R_t > 0$, x_t and g_t must be larger to offset y_t. For $R_t < 0$, x_t and g_t must be smaller. This discrepancy should reflect differently in the covariance of x_t and y_t under these conditions because x_t will vary more to make $R_t > 0$ compared to making $R_t < 0$, which contradicts equation (8).

As for the condition var ($x_t \mid R_t > 0$) = var ($x_t \mid R_t < 0$), it implies that the variability of x_t does not depend on whether R_t is positive or negative. However, given the conditions of R_t, the variability of x_t must adjust differently depending on R_t being positive or negative, due to the offset required by y_t and g_t, which also contradicts equation (8).

In sum, the assumptions cov (x_t, $y_t \mid R_t > 0$) = cov (x_t, $y_t \mid R_t < 0$) and var ($x_t \mid R_t > 0$) = var ($x_t \mid R_t < 0$) are inconsistent with $y_t < 0$, corr (x_t, y_t) > 0, corr (x_t, g_t) > 0, and corr (y_t, g_t) > 0. The reason is that the different necessary adjustments of x_t and g_t to maintain $R_t > 0$ or $R_t < 0$ cannot result in the same covariance and variance for x_t under the two different conditions of R_t. Thus, the set of conditions given is inherently inconsistent and problematic.

Lastly, the references they cite in their article are also inconsistent with their argument. In particular, Ball et al. (2013) assume that in their model, x_t, y_t, and g_t are stationary and time-independent random variables, and they cite Bachelier (1900), Samuelson (1965), Fama (1970), and Campbell, Lo, MacKinlay (1997) as references to support their assumption. However, these studies do not support their argument. Specifically, most of these studies show that stock prices do not follow the random walk pattern. Samuelson (1965) shows that, after some adjustments, future price follows martingale.[1] Martingales are not necessarily time-independent as shown in Appendix A, although their changes are uncorrelated (See Appendix B). Campbell et al. (1997) show the autocorrelation coefficients of stock prices are significantly different from 0, suggesting that price changes are not independent. Fama (1970, p. 392) states "…does not necessarily imply that the serial covariances of one-period returns are zero…. In the "fair game" efficient markets model (as defined by (4.1) and, (4.2)), the deviation of the return for t+1 from its conditional expectation is a "fair game" variable, but the conditional expectation itself can depend on the return observed for t." Again, this does not lend support to the time-independent assumption. According to the Fundamental Theorem of Asset Pricing, if S_t is the price process and B_t is the price of a risk-free asset, then S_t/B_t is a martingale under the risk-neutral measure. Once again, the changes in stock price are not necessarily independent from the viewpoint of modern finance theory. Note that stock price is not even martingale according to the theorem, let alone random walk. Please see Appendix C for a description of the Fundamental Theorem of Aset Pricing. We argue that Ball et al. (2013) do not recognize that these references are, in fact, not supportive of their argument. To end our discussion of Ball et al. (2013), we conclude that, from both the perspectives of inconsistent patterns between the stock return and the accounting earnings and inconsistent assumptions among several equations, the model provided by Ball et al. (2013) is actually wrong and puzzling.

Ball (2024) states "This concept of information as novelty, when applied to Fama's (1965) seminal framing of stock price behavior as a function of information arrival, leads to viewing stock price changes (i.e., returns) as independent across time, following so-called 'random walks' (Bachelier, 1900; Samuelson, 1965, 1973; Fama, 1970; Campbell et al., 1997)." This passage is largely derived from Ball et al. (2013), with an additional reference to Samuelson (1973). Instead of simply suggesting that stock price changes are time-independent, Ball (2024) asserts that stock prices follow random walks. This is quite remarkable, given that the finance and economic theories have long contested this view (Lucas, 1978; Harrison and Kreps, 1979; Harrison and Pliska, 1981; Delbaen and Schachermayer, 1994).

---

[1] We provide the details of martingale in Appendices A and B.

The research in economics and finance has challenged the randomness and independence of stock price movements. For instance, Lucas (1978) introduces the notion of rational expectations and equilibrium in asset markets, which implies that prices are not entirely random but are driven by rational agents' expectations based on all available information. Harrison and Kreps (1979) and Harrison and Pliska (1981) provide a rigorous mathematical framework for asset pricing under uncertainty. Moreover, Delbaen and Schachermayer (1994) formalize the Fundamental Theorem of Asset Pricing, which underlines that in an arbitrage-free market, there exists a risk-neutral measure under which the discounted asset prices are martingales. In particular, prices are sigma-martingales if there is no free lunch with vanishing risk (a no-arbitrage condition). This framework suggests that price movements are contingent upon the underlying economic conditions and constraints.

In conclusion, while Ball (2024) echoes the traditional view that stock prices follow random walks due to new information, the evolution of economic and financial theories has provided a more nuanced understanding of price dynamics, acknowledging the influence of broader economic variables and the limitations of the random walk hypothesis.

To sum up the first part, we discuss the three common definitions of earnings, critique several prior studies related to earnings quality, with the main focus on Dechow et al. (2010), and present our thoughts on the shortcomings mentioned. From our examination and discussion, it can be noticed that prior literature, even though published in top accounting journals, can be quite questionable and puzzling.

II. Definition of Auditing and Audit Quality

In the second section, we proceed to discuss the definition of auditing and audit quality, critique the shortcomings in the prior literature, with the main focus on DeFond and Zhang (2014), and present our opinions and thoughts on the shortcomings identified.

To begin with, the auditing standards 1001 (AS 1001) provides the following statement to describe the concept of auditing:

"*The objective of the ordinary audit of financial statements by the independent auditor is the expression of an opinion on the fairness with which they present, in all material respects, financial position, results of operations, and its cash flows in conformity with generally accepted accounting principles.*"

It is clear from the definition above that the goal of auditing is to express the extent to which the financial statements are prepared in conformity with generally accepted accounting principles. Recall the three definitions and measurement methods of earnings in the first section, among the three types of earnings, accounting earnings represent the earnings calculated in accordance with generally accepted accounting principles. Hence, following the definition of auditing stated in AS 1001, when the independent auditor audits the financial statements, he or she is going to verify whether the earnings presented in the financial statements are accounting earnings. If the earnings presented in the financial statements are not accounting earnings, the independent auditor should report them in the audit report.

After looking at the definition of auditing, we continue to comment on the work of DeFond and Zhang (2014). DeFond and Zhang (2014), a review article published in *Journal of Accounting and Economics*, provides a comprehensive review of audit quality. DeFond and Zhang (2014) discuss the demand for and the supply of audit services and the factors that potentially affect audit quality. In the abstract of the article, they provide the following definition of audit quality:

"*We define higher audit quality as greater assurance of high financial reporting quality.*"

Later in the article, DeFond and Zhang (2014) further explain their definition of audit quality as follows:

"*Accordingly, we define higher audit quality as greater assurance that the financial statements faithfully reflect the firm's underlying economics, conditioned on its financial reporting system and innate characteristics.*"

From these assertions, high financial reporting quality is defined as the financial statements faithfully reflecting the firm's underlying economics, conditioned on its financial reporting system and innate characteristics. However, this statement is inconsistent with the definition of auditing stated previously. According to AS 1001, the independent auditor's goal is to verify the extent to which the financial statements are prepared in conformity with generally accepted accounting principles, i.e., the accounting earnings, not the "underlying economics" or "economic earnings." Using the example in the first section, the accounting earnings and the economic earnings are $40,000 and $70,000, respectively. The former is calculated following generally accepted accounting principles, but the latter is not. Hence, even though the economic earnings are more reflective of the firm's underlying economics, the accounting earnings should be presented in the financial statements. Accordingly, the definition provided by DeFond and Zhang (2014) is questionable.

Regarding the definition of audit quality, DeFond and Zhang (2014) further criticize the definition of DeAngelo (1981) as follows:

"*Most studies define audit quality as some variation of "the market-assessed joint probability that a given auditor will both detect a breach in the client's accounting system,* *and report the breach"* (*e.g., DeAngelo, 1981*). *While this definition motivates a large body of research, it portrays auditing as a binary process, with the auditor's role reduced to the simple detection and reporting of "black and white" GAAP violations.*"

We argue that this statement is incorrect. Actually, the definition of DeAngelo (1981) does not imply that the auditor's role is a simple detection and reporting of "black and white" GAAP violations. Note that DeAngelo (1981) states that whether the auditor detects and reports the breach is a joint probability concept, with the probability being between zero and one. Hence, interpreting the definition of DeAngelo (1981) as a discrete "black and white" decision is inappropriate.

In addition to the questionable definition of audit quality, a much more severe shortcoming in DeFond and Zhang (2014) centers around their illustrations of the relationship between audit quality and financial reporting quality. DeFond and Zhang (2014) argue that the financial reporting quality of a firm is affected by not only audit quality but also the financial reporting system and the innate characteristics of the firm. In particular, they write down the following formula:

$$FRQ = f(AQ, R, I) \tag{9}$$

$$\frac{\partial FRQ}{\partial AQ} > 0 \tag{10}$$

Equation (9) states that the financial reporting quality of a firm (*FRQ*) is a function of audit quality (*AQ*), financial reporting system of the firm (*R*), and the innate characteristics of the firm (*I*). Equation (10) states that higher audit quality leads to higher financial reporting quality. Additionally, DeFond and Zhang (2014) analyze the relationship among *FRQ*, *AQ*, *R*, and *I*.

For the relationship among *FRQ*, *AQ*, and *I*, DeFond and Zhang (2014) analyze two types of firms. The first type consists of firms whose innate characteristics make them relatively hard to measure earnings; for example, firms with a lot of intangible assets. The other type consists of firms whose innate characteristics make them relatively easy to measure earnings; for example, firms whose assets are mostly tangible assets. DeFond and Zhang (2014) suggest that higher audit quality leads to higher financial reporting quality. However, the financial reporting quality is strictly lower for those firms whose innate characteristics make them hard to measure earnings. The firm's innate characteristics restrict the assured level of financial reporting quality that results from high audit quality.

For the relationship among *FRQ*, *AQ*, and *R*. DeFond and Zhang (2014) separate firms into two types: firms with high quality financial reporting system and firms with low quality financial reporting system. DeFond and Zhang (2014) suggest that, again, higher audit quality leads to higher financial reporting quality. However, the improvement is greater for those firms with low quality financial reporting system.

The discussion above seems reasonable. However, we argue that there is a critical issue omitted in the analysis. In equations (9) and (10), the analysis is generally unilateral. That is, financial reporting quality is affected by audit quality. We suggest the unilateral analysis of financial reporting quality and audit quality is incorrect. Audit quality also depends on the financial reporting quality of the firm simultaneously. For a firm with higher financial reporting quality, the auditor can ensure that the financial statements are prepared in conformity with generally accepted accounting principles with less effort. Hence, we argue that the relationship between financial reporting quality and audit quality is bidirectional instead of unilateral.[2]

Given that the relationship between financial reporting quality and audit quality is bidirectional, equation (10) does not hold as AQ is an endogenous variable. It is conceptually wrong to calculate the partial derivatives with respect to an endogenous variable in simultaneous equations.

We also argue that the discussions of the relationship among FRQ, AQ, R, and I are too simplified. On one hand, assuming the effect of audit quality on financial reporting quality is identical for all firms is questionable. The effect of audit quality on financial reporting quality can be different for firms with different innate characteristics. Furthermore, assuming a linear relationship may be unrealistic. The marginal effect of audit quality on financial reporting quality can also vary at different levels of audit quality.

To sum up the second part, we discuss the definition of auditing and comment on the work of DeFond and Zhang (2014). In particular, we first point out that the definition of audit quality provided by DeFond and Zhang (2014) is inappropriate because the purpose of auditing is to verify the extent to which the financial statements are prepared in conformity with generally accepted accounting principles instead of the underlying economics. Furthermore, we also highlight that the unilateral formulation of financial reporting quality and audit quality is incorrect. DeFond and Zhang (2014) aim to review the auditing literature and enhance our understanding of auditing and audit quality. However, there are some fundamental shortcomings in this review paper.

III. Concluding Remarks

In this paper, we aim to revisit the topics of earnings quality and audit quality, which are the two most crucial research topics in the fields of financial accounting and auditing.

---

2 We show that audit quality can be affected by financial reporting quality in the Appendix D. That is, we show that it can be true that $AQ = h(FRQ, R, I)$ for some function h.

We demonstrate that there are quite many shortcomings in the prior literature and present our views and thoughts on these shortcomings.

In the first part, we focus on the definitions of earnings and earnings quality. We critique the problems of Dechow et al. (2010) and related papers carefully. Specifically, we emphasize that the discussion in Dechow et al. (2010) is unclear and confusing. We also point out that the discussion on earnings persistence in the prior literature is wrong. Lastly, we show that the assumptions in the model of Ball et al. (2013) are inconsistent and problematic. We believe that all the issues we raise are important for future research.

As accounting standards such as International Financial Reporting Standards (IFRS) are getting more inclined to fair value measurement, we suggest that this movement is likely to make accounting earnings become more similar to economic earnings and improve earnings quality as economic earnings are more reflective of the changes of the value of the firm. For the measurement of earnings quality, we suggest that good proxies of earnings quality should possess the following characteristics: dynamic (adaptive to changes in the business environment and accounting practices) and predictive of future earnings.

In the second part, we focus on the definition of auditing and audit quality. We critique the fundamental problems of DeFond and Zhang (2014). In particular, we highlight that the definition of audit quality is problematic in DeFond and Zhang (2014). Additionally, we argue that, theoretically, the unilateral relationship between financial reporting quality and audit quality is incorrect. Previous studies usually regress the proxies of audit quality or financial reporting quality on Big 4 auditors to study the effect of Big 4 auditors on audit quality or financial reporting quality. From our discussion of bidirectional relationship, we cautious that this kind of research design is plagued with econometric issues.

Therefore, we suggest that other robust research designs should be considered. For example, recent advancements integrate IV methods with machine learning to improve causal inference (Mullainathan and Spiess, 2017; Chernozhukov, Chetverikov, Demirer, Duflo, Hansen, Newey, and Robins, 2018; Athey and Imbens, 2019). This approach can address endogeneity while allowing for flexible model specifications. Using DiD designs with continuous or multiple treatment variables allows researchers to account for heterogeneity in the impact of audit quality on financial reporting quality across different firms or contexts (e.g., Callaway and Sant'Anna, 2021). Incorporating dynamic panel data methods (e.g., system GMM) helps control for time-invariant unobserved heterogeneity and the potential feedback effects between audit quality and financial reporting quality. These newer methods offer more flexibility and robustness for empirical research on the relationship between financial reporting quality and audit quality, addressing the complexity of their interdependence while avoiding the limitations of traditional techniques (Arellano and Bover, 1995; Blundell and Bond, 1998).

The studies we critique in the paper are those published in top accounting journals (i.e., *Journal of Accounting and Economics*, *Journal of Accounting Research*, and *Contemporary Accounting Research*) and are those written by famous scholars in the fields of financial accounting and auditing. Nevertheless, we find that these studies are full of errors and problems. This reminds us that it is essential to evaluate the contexts in the papers carefully and think critically whether they are correct or not.

References

Arellano, M., and O. Bover. 1995. Another look at the instrumental variable estimation of error-components models. *Journal of Econometrics* 68 (1): 29-51.

Athey, S., and G. W. Imbens. 2019. Machine learning methods that economists should know about. *Annual Review of Economics* 11: 685-725.

Bachelier, L. 1900. Théorie de la spéculation. *Annales scientifiques de l'École Normale Supérieure* 17: 21-86.

Ball, R., S. P. Kothari, and V. V. Nikolaev. 2013. Econometrics of the Basu asymmetric timeliness coefficient and accounting conservatism. *Journal of Accounting Research* 51 (5): 1071-1097.

Ball, R. 2024. By what criteria do we evaluate accounting? Some thoughts on economic welfare and the archival literature. *Journal of Accounting Research* 62 (1): 7-54.

Blundell, R., and S. Bond. 1998. Initial conditions and moment restrictions in dynamic panel data models. *Journal of Econometrics* 87 (1): 115-143.

Callaway, B., and P. H. C. Sant'Anna. 2021. Difference-in-differences with multiple time periods. *Journal of Econometrics* 225 (2): 200-230.

Campbell, J. Y., A. W. Lo, and A. C. MacKinlay. 1997. *The Econometrics of Financial Markets.* Princeton, NJ: Princeton University Press.

Chernozhukov, V., D. Chetverikov, M. Demirer, E. Duflo, C. Hansen, W. Newey, and J. Robins. 2018. Double/debiased machine learning for treatment and structural parameters. *The Econometrics Journal* 21 (1): C1-C68.

DeAngelo, L. 1981. Auditor independence, 'low-balling' and disclosure regulation. *Journal of Accounting and Economics* 3 (2): 113-127.

Dechow, P., W. Ge, and C. Schrand. 2010. Understanding earnings quality: A review of the proxies, their determinants and their consequences. *Journal of Accounting and Economics* 50 (2-3): 344-401.

DeFond, M., and J. Zhang. 2014. A review of archival auditing research. *Journal of Accounting and Economics* 58 (2-3): 275-326.

Delbaen, F., and W. Schachermayer. 1994. A general version of the fundamental theorem of asset pricing. *Mathematische Annalen* 300: 463-520.

Fama, E. F. 1965. The behavior of stock-market prices. *The Journal of Business* 38 (1): 34-105.

Fama, E. F. 1970. Efficient capital markets: A review of theory and empirical work. *The Journal of Finance* 25 (2): 383-417.

Francis, J., P. Olsson, and K. Schipper. 2008. Earnings quality. *Foundations and Trends in*

Accounting 1 (4): 259-340.

Harrison, J. M., and D. M. Kreps. 1979. Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory* 20 (3): 381-408.

Harrison, J. M., and S. R. Pliska. 1981. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and Their Applications* 11 (3): 215-260.

Lucas, R. E. 1978. Asset prices in an exchange economy. *Econometrica* 46 (6): 1429-1445.

Mullainathan, S., and J. Spiess. 2017. Machine learning: An applied econometric approach. *Journal of Economic Perspectives* 31 (2): 87-106.

Penman, S. H., and T. Sougiannis. 1998. A comparison of dividend, cash flow, and earnings approaches to equity valuation. *Contemporary Accounting Research* 15 (3): 343-383.

Samuelson, P. A. 1965. Proof that properly anticipated prices fluctuate randomly. *Industrial Management Review* 6: 41-49.

Samuelson, P. A. 1973. Proof that properly discounted present values of assets vibrate randomly. *The Bell Journal of Economics and Management Science* 4 (2): 369-374.

Appendix A: Martingale

◆ Martingale and Independence of Increments

The differences (increments) of a martingale are not necessarily independent. We will explain what martingale is, the properties of increments, and the conditions under which these increments might exhibit independence.

◆ Martingale: Definition and Properties

A stochastic process $\{M_t\}$ is called a martingale with respect to a filtration $\{\boldsymbol{F}_t\}$ if it satisfies the following conditions:

1. **Adaptedness:** M_t is adapted to the filtration $\boldsymbol{F}_t$, meaning that M_t is measurable with respect to the information available up to time t.
2. **Integrability:** $\mathbf{E}[|M_t|] < \infty$ for all t.
3. **Martingale Property:** For all $s \geq 0$, $\mathbf{E}[M_{t+s} \mid \boldsymbol{F}_t] = M_t$.

◆ Increments of Martingale

The increments of a martingale are given by $M_{t+s} - M_t$. The martingale property implies that the expected value of the increment, given the past, is zero: $\mathbf{E}[M_{t+s} - M_t \mid \boldsymbol{F}_t] = 0$.

◆ Independence and Martingale Property

Independence of increments means that the value of one increment does not provide any information about the value of another increment. The martingale property requires that the future value, given the past, has an expected increment of zero, but it does not imply anything about the independence of increments.

◆ When Are Increments Independent?

While martingales do not generally have independent increments, there are special cases where this might be true. One such case is a **random walk**.

- **Simple Symmetric Random Walk:** Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables taking values ±1 with equal probability. Define $M_t = \sum_{i=1}^{t} X_i$. This is a martingale with independent increments because each X_i is independent of the others. In general, for a martingale $\{M_t\}$, the increments $M_{t+s} - M_t$ need not be independent. They are typically dependent because the value of M_t at time t can influence the future values M_{t+s}.

◆ Example: Brownian Motion

Consider a standard Brownian motion $\{W_t\}$:

- W_t is a martingale with respect to its natural filtration.
- W_t has independent increments: $W_{t+s} - W_t$ is independent of $\boldsymbol{F}_t$ (the history up to time t) and is normally distributed with mean 0 and variance s.

In this case, the martingale (Brownian motion) has independent increments, but this is a specific property of Brownian motion and not a general property of all martingales.

◆ Martingale and Martingale Differences

- **Martingale:** The increments $M_{t+s} - M_t$ of a martingale $\{M_t\}$ have an expected value of zero given the past, but they are not generally independent.
- **Independence:** Independence of increments is a stronger condition that does not follow from the martingale property alone.
- **Special Cases:** Some processes, such as the simple symmetric random walk or Brownian motion, have both the martingale property and independent increments, but these are specific examples.

◆ Example: A Martingale with Dependent Increments

Consider the following process:

- Let X_i be a random variable that takes values ±1 with equal probability 0.5.
- For $n \geq 2$, define $X_n = X_{n-1} \cdot Z_n$, where Z_n is an i.i.d. random variable that takes values ±1 with equal probability.

Now define the martingale $\{M_n\}$ as:

$$M_n = \sum_{i=1}^{n} X_i$$

◆ Martingale Property

To show that $\{M_n\}$ is a martingale with respect to its natural filtration $\{\boldsymbol{F}_n\}$, where $\boldsymbol{F}_n = \sigma(X_1, X_2, \ldots, X_n)$:

1. Adaptedness: M_n is adapted to $\boldsymbol{F}_n$ because it is constructed from $X_1, X_2, \ldots, X_n$.

2. Integrability: $\mathbf{E}[|M_n|] < \infty$ for all n because X_i are finite-valued random variables.

3. Martingale Property: We need to show that $\mathbf{E}[M_{n+1}| \boldsymbol{F}_n] = M_n$.

To check the martingale property:

$$M_{n+1} = M_n + X_{n+1}$$

Given the definition of X_{n+1}, we have:

$$\mathbf{E}[M_{n+1}| \boldsymbol{F}_n] = M_n + \mathbf{E}[X_{n+1}| \boldsymbol{F}_n]$$

Since $X_{n+1} = X_n \cdot Z_{n+1}$ and Z_{n+1} is independent of $\boldsymbol{F}_n$ with $\mathbf{E}[Z_{n+1}] = 0$:

$$\mathbf{E}[X_{n+1} | \boldsymbol{F}_n] = \mathbf{E}[X_n \cdot Z_{n+1} | \boldsymbol{F}_n] = X_n \cdot \mathbf{E}[Z_{n+1} | \boldsymbol{F}_n] = X_n \cdot 0 = 0.$$

Thus,

$$\mathbf{E}[M_{n+1} | \boldsymbol{F}_n] = M_n + 0 = M_n.$$

Therefore, $\{M_n\}$ is a martingale.

- **Dependence of Increments**

To see that the increments are not independent, consider:

$$M_1 = X_1,$$
$$M_2 = X_1 + X_2,$$
$$M_3 = X_1 + X_2 + X_3,$$
$$M_4 = X_1 + X_2 + X_3 + X_4.$$

Since $X_2 = X_1 \cdot Z_2$ and $X_3 = X_2 \cdot Z_3 = X_1 \cdot Z_2 \cdot Z_3$, the values of $X_2, X_3, \ldots$ depend on the previous X_i. Therefore, the increments $M_{n+1} - M_n = X_{n+1}$ are not independent because X_{n+1} depends on X_n, which in turn depends on $X_{n-1}, \ldots, X_1$.

Appendix B: Proof of Martingale Increments are Uncorrelated

1. Martingale:

A sequence $\{X_n\}$ is a martingale with respect to a filtration $\{\boldsymbol{F}_n\}$ if

- X_n is $\boldsymbol{F}_n$-measurable.
- $\mathbf{E}[|X_n|] < \infty$.
- $\mathbf{E}[X_{n+1} \mid \boldsymbol{F}_n] = X_n$.

2. Martingale Increment:

The increment D_n of a martingale $\{X_n\}$ is defined as

$$D_n = X_n - X_{n-1}.$$

3. Uncorrelated Increments:

We aim to show that $\mathbf{E}[D_nD_m] = 0$ for $n \neq m$.

◆ **Proof**

1. Martingale Difference Property:

We need to show that $\mathbf{E}[D_n \mid \boldsymbol{F}_{n-1}] = 0$.
By definition:

$$D_n = X_n - X_{n-1}.$$

Given that $\{X_n\}$ is a martingale, we have:

$$\mathbf{E}[X_n \mid \boldsymbol{F}_{n-1}] = X_{n-1}.$$

Therefore, $\mathbf{E}[D_n \mid \boldsymbol{F}_{n-1}] = \mathbf{E}[X_n - X_{n-1} \mid \boldsymbol{F}_{n-1}] = \mathbf{E}[X_n \mid \boldsymbol{F}_{n-1}] - X_{n-1} = 0$.

2. Expectation of the Product of Increments:

We need to compute $\mathbf{E}[D_nD_m] = 0$ for $n \neq m$. Without loss of generality, assume that $n < m$. Consider $\mathbf{E}[D_nD_m] = \mathbf{E}[(X_n - X_{n-1})(X_m - X_{m-1})]$.

3. Expanding the Product:

$\mathbf{E}[D_nD_m] = \mathbf{E}[X_nX_m] - \mathbf{E}[X_nX_{m-1}] - \mathbf{E}[X_{n-1}X_m] + \mathbf{E}[X_{n-1}X_{m-1}]$

4. Using the Martingale Property:

Since X_n is $\boldsymbol{F}_n$-measurable and $X_n - X_{n-1}$ is the increment starting from n, we can use the fact that $\mathbf{E}[X_n \mid \boldsymbol{F}_{n-1}] = X_{n-1}$:

$$\mathbf{E}[X_nX_m \mid \boldsymbol{F}_{n-1}] = X_{n-1}\mathbf{E}[X_m \mid \boldsymbol{F}_{n-1}]$$

5. Expectation Conditioned on Fn:

For $n < m$, X_m is independent of $\boldsymbol{F}_n$ given the martingale property, and $\mathbf{E}[X_m \mid \boldsymbol{F}_n] = X_n$, we have $\mathbf{E}[X_n X_m] = \mathbf{E}[X_n\, \mathbf{E}\, [X_m \mid \boldsymbol{F}_n]] = \mathbf{E}[X_n X_n] = \mathbf{E}[X_n^2]$. Similarly, we have $\mathbf{E}[X_n X_{m-1}] = \mathbf{E}[X_n X_n] = \mathbf{E}[X_n^2]$, $\mathbf{E}[X_{n-1} X_m] = \mathbf{E}[X_{n-1} X_{n-1}] = \mathbf{E}[X_{n-1}^2]$, and $\mathbf{E}[X_{n-1} X_{m-1}] = \mathbf{E}[X_{n-1} X_{n-1}] = \mathbf{E}[X_{n-1}^2]$.

6. Summarizing the Result:

Since increments D_n are zero mean, $D_n = X_n - X_{n-1}$ are uncorrelated:

$\mathbf{E}[D_n D_m] = \mathbf{E}[D_n]\mathbf{E}[D_m] = 0$.

Appendix C: The Fundamental Theorem of Asset Pricing (FTAP)

- **Statement of the Theorem** (Note that a formal and rigorous statement involves more mathematics, we choose to present it in a simple way.)

No-Arbitrage Condition: If a financial market is arbitrage-free, then there exists at least one equivalent martingale measure.
Completeness and Uniqueness: If a financial market is arbitrage-free and complete, then there exists a unique equivalent martingale measure.

- **Arbitrage**

Arbitrage is the practice of exploiting price differences between different markets or forms of an asset to make a profit with no risk and no net investment. An arbitrage opportunity exists if one can construct a portfolio with no initial cost that guarantees a risk-free profit in the future.

- **Equivalent Martingale Measure (Risk-Neutral Measure)**

A probability measure Q is called an equivalent martingale measure if it is equivalent to the real-world probability measure P and under Q, the discounted price process of any traded asset is a martingale. This means:

$$E^Q = \left[\frac{S_{t+1}}{B_{t+1}} \middle| F_t \right] = \frac{S_t}{B_t} ,$$

where S_t is the price process and B_t is the price of a risk-free asset. The FTAP underlies various financial models, including the Black-Scholes model for option pricing.

Appendix D: Proof that $AQ = h(FRQ, R, I)$

- To determine whether $AQ = h(FRQ, R, I)$ for some function h, we need to explore the relationships given

1. $FRQ = f(AQ, R, I)$, and
2. $AQ = g(R, I)$.

- **Case Analysis**

Case 1: Independent Relationships

If $FRQ = f(AQ, R, I)$ and $AQ = g(R, I)$ are independent, meaning that *FRQ* and *AQ* are related to *R* and *I* in ways that do not depend on each other, then it is straightforward to see that *AQ* is solely a function of *R* and *I*: $AQ = g(R, I)$. Here, *FRQ* is determined by *f* using the given *AQ*, *R*, and *I*, but *FRQ* does not influence *AQ*. Thus, *AQ* remains a function of only *R* and *I*.

Case 2: Interdependent Relationships

If the relationships are interdependent, we need to analyze whether *FRQ* can affect *AQ*. Given $AQ = g(R, I)$, we can substitute *AQ* into $FRQ = f(AQ, R, I)$ as $FRQ = f(g(R, I), R, I)$. In this case, *FRQ* is a function of *R* and *I* via *g*. To express *AQ* as a function of *FRQ*, *R*, and *I*, we consider the possibility of inverting the relationship $FRQ = f(AQ, R, I)$. If *f* is invertible with respect to *AQ*, we can find *AQ* as $AQ = f^{-1}(FRQ, R, I)$. Thus, there exists an *h* such that $AQ = h(FRQ, R, I)$, where $h(FRQ, R, I) = f^{-1}(FRQ, R, I)$.

- **Conclusion**

If *f* is invertible with respect to *AQ*, then there exists a function *h* such that $AQ = h(FRQ, R, I)$. If *f* is not invertible with respect to *AQ*, then we cannot express *AQ* as a function of *FRQ*, *R*, and *I*. Therefore, the existence of *h* depends on the invertibility of *f* with respect to *AQ*. If *f* is invertible, such a function *h* exists.

- **Showing that a Non-Invertible Function is Impossible**

To demonstrate that a function that is not invertible is impossible (with measure zero), we use concepts from measure theory and analysis, particularly Sard's Theorem.

- **Definitions and Concepts**

1. Invertibility: A function $f : A \to B$ is invertible if there exists a function $f^{-1} : B \to A$ such that $f^{-1}(f(x)) = x$ for all $x \in A$ and $f(f^{-1}(y)) = y$ for all $y \in B$.
2. Measure Zero: A set $S \subset \mathbf{R}^n$ has measure zero if, for every $\varepsilon > 0$, there exists a countable

collection of n-dimensional intervals (or cubes) $\{I_k\}$ such that $S \subset \bigcup_{k=1}^{\infty} I_k$ and the sum of the volumes of these intervals is less than ε.

◆ Sard's Theorem

If $f : \mathbf{R}^n \to \mathbf{R}^m$ $(n \geq m)$ is a smooth function (i.e., C^∞), then the set of critical values of f (the image of the set of critical points) has Lebesgue measure zero in $\mathbf{R}^m$. Here, a critical point is a point where the Jacobian matrix Df does not have full rank.

◆ Application to Invertibility

1. **Non-Invertibility and Critical Points:** A function f is not locally invertible at a point x if the Jacobian matrix $Df(x)$ is not invertible (i.e., does not have full rank).
2. **Measure Zero of Non-Invertible Points:** According to Sard's Theorem, the set of critical values of f, which includes points where f is not locally invertible, has measure 0.

◆ Proof Outline

1. **Smooth Function:** Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a smooth function. The Jacobian matrix Df at each point $x \in \mathbf{R}^n$ is an $n \times n$ matrix.
2. **Critical Points:** A point $x \in \mathbf{R}^n$ is a critical point if the Jacobian matrix $Df(x)$ does not have full rank. The set of critical points $C \subset \mathbf{R}^n$ is the set where f fails to be locally invertible.
3. **Sard's Theorem:** Apply Sard's Theorem to f. The image of the critical set $f(C)$ has Lebesgue measure 0 in $\mathbf{R}^n$.
4. **Measure Zero:** The set of points in the codomain where f fails to be invertible (because their preimages contain critical points) has measure 0.

◆ General Proof: Function Almost Everywhere Invertible

To show that a non-differentiable function can be almost everywhere invertible, we will use measure theory principles and properties of continuous, strictly monotonic functions.

◆ Key Theorems and Concepts

1. **Lebesgue's Differentiation Theorem:** This theorem states that a monotonic function on $\mathbf{R}$ is differentiable almost everywhere. The set of points where the function is not differentiable has measure zero.
2. **Properties of Monotonic Functions:** A continuous, strictly monotonic function on $\mathbf{R}$ is injective and thus invertible on its range.

◆ Proof Outline

1. **Strict Monotonicity and Injectivity:** Discuss how strict monotonicity ensures injectivity.
2. **Measure Zero of Non-Invertible Points:** Use Lebesgue's Differentiation Theorem to show that the set of non-differentiable points has measure zero.
3. **Invertibility on Full Measure:** Argue that outside the measure-zero set, the function is invertible.

Detailed Proof

Step 1: **Strict Monotonicity and Injectivity**

Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous, strictly monotonic function.

- **Strict Monotonicity:** If f is strictly increasing, for any $x_1, x_2 \in \mathbf{R}$ with $x_1 < x_2$, we have $f(x_1) < f(x_2)$.
- **Injectivity:** Because f is strictly monotonic, it is injective (one-to-one).

Step 2: **Measure Zero of Non-Invertible Points**

- **Non-Invertible Points:** For f to be invertible, it needs to be locally injective, which typically fails where the function is not differentiable.
- **Lebesgue's Differentiation Theorem:** This theorem states that a monotonic function on $\mathbf{R}$ is differentiable almost everywhere. Thus, the set $N \subset \mathbf{R}$ where f is not differentiable has Lebesgue measure zero.

Step 3: **Invertibility on Full Measure**

Since f is differentiable almost everywhere, it is locally invertible almost everywhere. The set of non-invertible points N has measure zero. Therefore, f is invertible almost everywhere in $\mathbf{R} \setminus N$.

Conclusion

For a continuous, strictly monotonic function $f : \mathbf{R} \to \mathbf{R}$, the set of points where f fails to be invertible (non-differentiable points) has measure zero. Therefore, f is almost everywhere invertible.